Lévy flights, Lévy semigroups and fractional quantum mechanics

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Probability density function dynamics and its heavy-tailed asymptotics (invariant pdf) **induced by:**

- **Lévy flights** - *hint:* gradient perturbations of symmetric stable and (quasi-) relativistic noise generators

- **Lévy-Schrödinger semigroups** - *hints:* additive perturbations of nonlocal noise generators, Schrödinger's boundary data problem

- **Wiener noise** - *hints:* specific gradient perturbations of the Laplacian, logarithmic potentials, heavy-tailed asymptotics of diffusion processes

- **Pseudo-differential QM** - *hints:* analytic continuation in time, holomorfic semigroups, unitary dynamics, pseudo-differential spectral problems, ground state induced jump-type process

Gradient perturbations of noise generators: standards

J. Math. Phys. (1999), (2000)

It is worth noting that when the transition function is stochastically continuous (see Sec. IV B), then the corresponding semigroup T_t in $C_0(\mathbf{R})$ defined by

$$(T_t f)(x) = \int_{-\infty}^{\infty} p_t(y|x) f(y) dy \tag{21}$$

is strongly continuous, and so its generator L is densely defined.

In such a case we can also define an adjoint semigroup T_t^* acting on the space of (probability) densities $L^1(\mathbf{R}, dx)$,

$$(T_t^* \rho)(u) = \int_{-\infty}^{\infty} p_t(u|v) \rho(v) dv. \tag{22}$$

Its generator we denote by L^*.

$$L = L_0 + b\nabla \qquad\qquad L^* = L_0 - \nabla(b\ \cdot)$$

transition probability function of the process $\mathbf{u}(t)$ satisfies the backward equation

$$\frac{\partial p_t(u|v)}{\partial t} = L_0 p_t(u|\cdot)(v) + b(v)\nabla_v p_t(u|v)$$

First Kolmogorov eq.

and the forward equation (the Fokker–Planck equation analog)

$$\frac{\partial p_t(u|v)}{\partial t} = L_0 p_t(\cdot|v)(u) - \nabla_u[b(u) p_t(u|v)].$$

Second Kolmogorov eq.

This equation determines the time evolution of pdfs (given, the initial data)

Brownian (Wiener noise) detour

Gradient perturbation of $L_0 = D\,\Delta$ $\quad L = D\,\Delta + b\nabla$ $\quad$ Diffusion generator

$L^* = D\,\Delta - \nabla(b\,\cdot)$ $\qquad \partial_t \rho = D\triangle\rho - \nabla\,(b\cdot\rho)\,.$ $\quad$ Fokker-Planck equation

stationary asymptotic regime $\qquad \rho(x,t) \to \rho_*(x) \Longleftrightarrow b(x) = D\nabla \ln \rho_*$

make an ansatz $\Psi(x,t) = \rho(x,t)\,\rho_*^{-1/2}(x)$

$$\exp(-t\hat{H}/2mD)\Psi_0 = \Psi \;\Longrightarrow\; \partial_t \Psi = \left[D\Delta - \frac{V}{2mD}\right]\Psi = -\frac{\hat{H}}{2mD}\,\Psi$$

(Schrödinger) semigroup $\exp(-t\hat{H}/2mD)$

(Additive perturbation of the Laplacian)

$\hat{H} \geq 0$ with bottom eigenvalue 0

$$V = 2mD^2\,\frac{\Delta\rho_*^{1/2}}{\rho_*^{1/2}}$$

Another form of the compatibility condition

$$V(x) = 2mD^2\left[\frac{b^2}{2D} + \nabla b\right]$$

Typical values of D: $\quad D = \hbar/2m \qquad D = k_B T/m\beta$

Semigroup kernel vs transition density

Note: suitable restrictons upon the semigroup potential need to be respected, to have a positive and continuous semigroup kernel function

$$k(y,s,x,t) = \left(\exp[-(t-s)\hat{H}]\right)(y,x) \quad = \int exp[-\int_s^t \mathcal{V}(X(u),u)du]\, d\mu[s,y \mid t,x]$$

We can relate the semigroup kernel and the transition density by means of Doob's type multiplicative transformation

Re: Doob's transformation. For given $X(t)$ let L be such that $Lh = 0$, $h > 0$. Then $X_h(t)$ is generated by $L_h = h^{-1}Lh \equiv L_h g = h^{-1}L(hg))$

$$k(y,s,x,t) = p(y,s,x,t)\frac{\rho_*^{1/2}(y)}{\rho_*^{1/2}(x)}$$

$$\rho(x,t) \doteq \int p(y,s,x,t)\rho(y,s)dy$$

$$\partial_t \rho = D\triangle\rho - \nabla\,(b\cdot\rho)$$

Gradient and additive perturbations of the **Laplacian** induce **the same** pattern of dynamical behavior of the pdf $\rho(x,t)$

Eigenfunction expansions – the classics: heat kernel and Mehler's kernel

For clarity of discussion, it is instructive to invoke explicit examples. We pass to one spatial dimension and rescale (or completely scale away) a diffusion coefficient. Given a spectral solution for $\hat{H} = -\Delta + V \geq 0$, the integral kernel of $\exp(-t\hat{H})$ reads

$$k(y,x,t) = k(x,y,t) = \sum_j \exp(-\epsilon_j t)\, \Phi_j(y)\Phi_j^*(x).$$

we assume $\epsilon_0 = 0$ and the sum may be replaced by an integral in case of a continuous spectrum

Set $V(x) = 0$ identically. Then we end up with the familiar heat kernel

$$k(y,x,t) = [\exp(t\Delta)](y,x) = (2\pi)^{-1/2} \int \exp(-p^2 t)\, \exp(ip(y-x)\, dp =$$

$$(4\pi t)^{-1/2} \exp[-(y-x)^2/4t]$$

Consider $\hat{H} = (1/2)(-\Delta + x^2 - 1)$

$$k(y,x,t) = k(x,y,t) = [\exp(-t\hat{H})(y,x) =$$

$$[\pi(1-\exp(-2t))^{-1/2} \exp[-(1/2)(x^2-y^2) - (1-\exp(-2t))^{-1}\,(x\exp(-t)-y)^2]$$

$\int k(y,x,t)\exp[(y^2-x^2)/2]\, dy = 1$ actually defines a transition probability density $p(y,0,x,t) \equiv p(y,x,t) = k(y,x,t)\, \rho_*^{1/2}(x)/\rho_*^{1/2}(y)$

$$p(y,0,x,t) \equiv p_t(x|y) = [\pi(1-e^{-2t})]^{-1/2} \exp\left[-\frac{(x-e^{-t}y)^2}{(1-e^{-2\beta t})}\right]$$

$L = b(y)\,\nabla_y + (1/2)\Delta_y,\ b(y) = -y$ Ornstein-Uhlenbeck process $L^* = (1/2)\Delta_x - \nabla_x[b(x)\,\cdot]$ and $b(x) = -x$

Wherein Lévy-Schrödinger semigroups ?

First input: Schrödinger's boundary data problem (1932)

Deduce the Markovian interpolation consistent with a given pair of boundary measure data at fixed initial and terminal time instants $t_1 < t_2$; A and B are two Borel sets in R.

$$m(A, B) = \int_A dx \int_B dy \, m(x, y),$$

$$\int_R m(x, y) dy = \rho(x, t_1),$$

$$\int_R m(x, y) dx = \rho(y, t_2),$$

where

$$m(x, y) = f(x) k(x, t_1, y, t_2) g(y)$$

$f(x)$ and $g(y)$ are of the same sign and nonzero, $k(x, s, y, t)$ is an a priori chosen, bounded strictly positive and continuous (dynamical semigroup) kernel, $t_1 \leq s < t \leq t_2$.

Bernstein vs Markovian, c.f. J-C. Zambrini (1986-7)

Prescribing $k(x,s,y,t)$ in advance, we have functions $f(x), g(y)$ determined uniquely (up to constant factors) ba marginal data, c.f. Beurling, Fortet, Jamison.

By denoting

$$\theta_*(x,t) = \int f(z)k(t_1,z,x,t)dz$$

$$\theta(x,t) = \int k(x,t,z,t_2)g(z)dz$$

it follows that

$$\rho(x,t) = \theta(x,t)\theta_*(x,t) = \int p(y,s,x,t)\rho(y,s)dy,$$

$$p(y,s,x,t) = k(y,s,x,t)\frac{\theta(x,t)}{\theta(y,s)},$$

$$t_1 \leq s < t \leq t_2$$

Note: If we assume that $g(x) = \rho_*^{1/2}(x)$, then likewise $\theta(x) = \rho_*^{1/2}(x)$, so we end up with previously mentioned Doob's type mapping of a semigroup kernel into a transition density.

Useful concept, to be exploited later:

Targeted stochasticity as a specialized version of the Schrödinger boundary data problem: **given a predefined pdf** ρ_* ; ask whether it can be interpreted as a unique **asymptotic invariant** pdf for each meber in the variety of inequivalent Markovian processes (to be pre-selected as well) ?

Second input: elementary harmonic/functional analysis

Let us consider self-adjoint operators (Hamiltonians) with dense domains in $L^2(R)$, of the form $\hat{H} = F(\hat{p})$, where $\hat{p} = -i\nabla$ and for $-\infty < k < +\infty$, $F = F(k)$ is a real valued, bounded from below, locally integrable function. For $t \geq 0$ we have:

$$exp(-t\hat{H}) = \int_{-\infty}^{+\infty} exp[-tF(k)]dE(k)$$

$dE(k)$ is the spectral measure of $\hat{p}$.

Let us set

$$k_t = \frac{1}{\sqrt{2\pi}}[exp(-tF(p)]^{\vee}$$

then the action of $exp(-t\hat{H})$ can be given in terms of a convolution: $exp(-t\hat{H})f = f * k_t$, where $(f * g)(x) := \int_R g(x-z)f(z)dz$.

If $F(p)$ satisfies the Lévy-Khintchine formula, then k_t is a positive measure for all $t \geq 0$ and we arrive at the simplest (free noise) positivity preserving semigroups.

The integral part of the L-K formula is responsible for random jumps ($\nu(dy)$ stands for the Lévy measure):

$$F(p) = -\int_{-\infty}^{+\infty} [exp(ipy) - 1 - \frac{ipy}{1+y^2}]\nu(dy)$$

Third input: (pseudo) relativistic Hamiltonians

$$F_0(p) = |p|$$

$$F_m(p) = \sqrt{p^2 + m^2} - m\,, \,, m > 0$$

(better known as $H_{cl} = \sqrt{m^2c^4 + c^2p^2} - mc^2$)
Within the ramifications of the Schrödinger boundary data problem set $\theta(x,t) \equiv 1$ and $\theta_*(x,t) \doteq \rho(x,t)$ so that

$$[exp(-t\hat{H})\rho](x) = \rho(x,t)$$

where $F(p \to -i\nabla) := \hat{H}$ implies

$$F_0(p) \Longrightarrow \partial_t \rho(x,t) = -|\nabla|\rho(x,t)$$

$$F_m(p) \Longrightarrow \partial_t \rho(x,t) = -[\sqrt{-\Delta + m^2} - m]\rho(x,t)$$

F_0 is a special (Cauchy) case of the symmetric stable probability laws and readily generalizes to (we can parallel this step by a lift from R to R^n)

$$F_\mu = |p|^\mu \to \partial_t \rho(x,t) = -|\Delta|^{\mu/2}\rho(x,t)$$

with $0 < \mu < 2$. (Note: $\partial_t \rho = \Delta\rho$ derives from the Wiener process and $\hat{H} = -\Delta$)

$$(-\Delta)^{\mu/2} f(x) = |\Delta|^{\mu/2} f(x) = |\nabla|^{\mu} f(x) = -\frac{\Gamma(1+\mu)\,\sin(\pi\mu/2)}{\pi} \int \frac{f(y)-f(x)}{|x-y|^{1+\mu}}\,dy$$

$0 < \mu < 2$ and the integral is interpreted in terms of the Cauchy principal value.

n-dimensional ($\mathrm{x} \in R^n$) generalization of the stable generator

$$|\Delta|^{\mu/2} f(\mathrm{x}) = -\frac{2^{\mu}\,\Gamma(\frac{\mu+n}{2})}{\pi^{n/2}|\Gamma(-\frac{\mu}{2})|} \int \frac{f(\mathrm{y})-f(\mathrm{x})}{|\mathrm{x}-\mathrm{y}|^{\mu+n}}\,d^n y = -\int [f(\mathrm{x}+\mathrm{y}) - f(\mathrm{x}]\,\nu_{\mu}(d\mathrm{y})$$

$\nu_{\mu}(d\mathrm{y})$ stands for a (self-defining) Lévy measure $\sim 1/|\mathrm{y}|^{\mu+n}$

Additive perturbation via Doobs'-type transformation

Vilela Mendes (1996)
Brockmann/Sokolov (2002)

$$\partial_t \rho = -|\Delta|^{\mu/2} \rho = \int [w(x|z)\rho(z) - w(z|x)\rho(x)]dz$$

$w(x|y) = w(x|y) \sim 1/|x-y|^{1+\mu}$ replace by $w_{\phi}(x|y) \sim \dfrac{\exp[\Phi(x) - \Phi(y)]}{|x-y|^{1+\mu}}$

$$\partial_t \rho = -|\Delta|_{\Phi}^{\mu/2} f = -\exp(\Phi)\,|\Delta|^{\mu/2}[\exp(-\Phi)\rho] + \rho\exp(-\Phi)|\Delta|^{\mu/2}\exp(\Phi)$$

$\rho(x,t) \to \rho_*(x) = \exp(2\Phi)$ If a contractive semigroup $\exp(-t\hat{H})$ is involved !

$$\partial_t \Psi = -\hat{H}_{\mu}\Psi$$

$$\hat{H}_{\mu} \equiv |\Delta|^{\mu/2} + \mathcal{V}(x) \qquad \mathcal{V} = -\frac{|\Delta|^{\mu/2}\rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\rho(x,t) = \Psi(x,t)\,\rho_*^{1/2}(x)$$

„Rough” conceptual guide: Cauchy semigroup

J. Math. Phys. (1999)

$$\partial_t \theta_* = -|\nabla|\theta_* - V\theta_* , \qquad \partial_t \theta = |\nabla|\theta + V\theta, \tag{21}$$

where V is a measurable function such that:

(a) for all $x \in R$, $V(x) \geqslant 0$,
(b) for each compact set $K \subset R$ there exists C_K such that for all $x \in K$, V is locally bounded $V(x) \leqslant C_K$.

Lemma 5: If $1 \leqslant r \leqslant p \leqslant \infty$ and $t > 0$, then the operators T_t^V defined by

$$(T_t^V f)(x) = E_x^C \left\{ f(X_t^C) \exp\left[-\int_0^t V(X_s^C) ds \right] \right\}$$

are bounded from $L^r(R)$ into $L^p(R)$. Moreover, for each $r \in [1,\infty]$ and $f \in L^r(R)$, $T_t^V f$ is a bounded and continuous function.

Lemma 7: For any $p \in [1,\infty]$ and $f \in L^p(R)$ there holds

$$(T_t^V f)(x) = \int_R k_t^V(x,y) f(y) dy, \qquad \text{where } k_t^V(x,y) \geqslant 0 \text{ almost everywhere}$$

Lemma 8: $k_t^V(x,y)$ is jointly continuous in (x,y).

Lemma 9: $k_t^V(x,y)$ is strictly positive.

let $\rho_0(x)$ and $\rho_T(x)$ be strictly

positive densities. Then, the Markov process X_t^V characterized by the transition probability density:

$$p^V(y,s,x,t)=k_{t-s}^V(x,y)\,\frac{\theta(x,t)}{\theta(y,s)} \tag{23}$$

and the density of distributions

$$\rho(x,t)=\theta_*(x,t)\theta(x,t),$$

where

$$\theta_*(x,t)=\int_R k_t^V(x,y)f(y)dy, \qquad \theta_*(y,t)=\int_R k_{T-t}^V(x,y)g(x)dx$$

is precisely that interpolating Markov process to which Theorem 1 extends its validity, when the perturbed semigroup kernel replaces the Cauchy kernel.

Clearly, for all $0\leqslant s\leqslant t\leqslant T$ we have

$$\theta_*(x,t)=\int_R k_{t-s}^V(x,y)\theta_*(y,s)dy, \qquad \theta(y,s)=\int_R k_{t-s}^V(x,y)\theta(x,t)dx \tag{24}$$

Association: set $\theta_*=\Psi$, $\theta=\rho_*^{1/2}$, so getting $\rho(x,t)=(\theta\theta_*)(x,t)=\Psi(x,t)\rho_*^{1/2}(x)$ and $\partial_t\Psi=-\hat{H}\Psi$ with $\hat{H}=|\nabla|+V$

Response to external potentials (physicist's view)

Lévy-Schrödinger semigroups

$$\hat{H}_\mu \doteq \lambda|\Delta|^{\mu/2} + \mathcal{V}$$

$$\exp(-t\hat{H}_\mu)$$

Schrödinger's boundary data problem

$$\partial_t \theta_* = -\lambda|\Delta|^{\mu/2}\theta_* - \mathcal{V}\theta_* .$$

$$\partial_t \theta = \lambda|\Delta|^{\mu/2}\theta + \mathcal{V}\theta$$

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2}\, \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\theta_*(x,t) = \rho(x,t)\, \exp[-\Phi(x)]$$

$$\theta(x) = \exp[\Phi(x)] = \rho_*^{1/2}(x) \qquad \theta^*(x,t)\theta(x,t) = \rho(x,t)$$

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda(\exp\Phi)|\Delta|^{\mu/2}[\exp(-\Phi)\rho] - \mathcal{V}\cdot\rho$$

$$\partial_t \rho = -\exp(\Phi)\, |\Delta|^{\mu/2}[\exp(-\Phi)\rho] + \rho\exp(-\Phi)|\Delta|^{\mu/2}\exp(\Phi)$$

Langevin (SDE) scenario

$$\dot{x} = b(x) + A^\mu(t) \Longrightarrow \partial_t \rho = -\nabla(b\cdot\rho) - \lambda|\Delta|^{\mu/2}\rho$$

Targeted stochasticity

$$\rho(x,t) \to \rho_*(x)$$

$$b(x) = -\lambda \frac{\int |\Delta|^{\mu/2}\rho_*(x)\, dx}{\rho_*(x)}$$

$$\partial_t \rho = -\nabla(b\cdot\rho) - \lambda|\Delta|^{\mu/2}\rho$$

Incompatible !

c.f. Doob's h-transform

Pictorial intuitions (physics) concerning **Lèvy semigroups:**
Lèvy processes in inhomogeneous media and thermal equilibrium

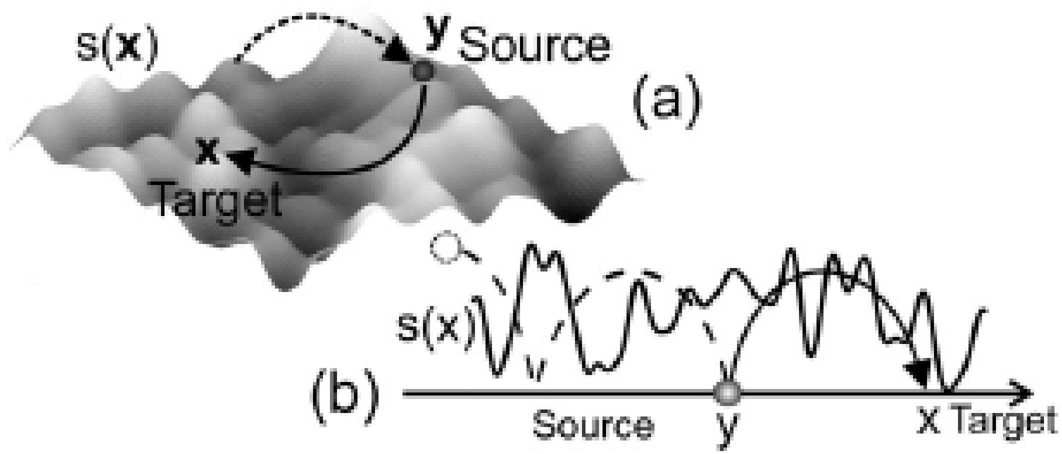


Figure 1. Random walk processes in inhomogeneous salience fields $s(x)$ in two (a) and one (b) dimensions. Source and target locations of a random jump are denoted by y and x, respectively.

The Belik and Brockmann (2007) attractivity or salience field s(x) is identical with our invariant pdf $\rho_*(x)$, while in the explicit Boltzmann-Gibbs form !

$$\rho_*(x) = C\exp(-\lambda V(x)) = \exp(2\Phi)$$

$$\lambda = (k_B T)^{-1} \ (k_B \text{ is Boltzmann constant})$$

Gibbs-Boltzmann equilibria **are incompatible** with the Langevin (SDE) modeling of Lèvy flights

Cauchy driver: invariant density and stochastic targeting

Ornstein-Uhlenbeck-Cauchy process

$$\partial_t \rho = -\lambda |\nabla| \rho + \nabla [(\gamma x) \rho]$$

$$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

Cauchy semigroup with the same (Cauchy pdf) target

$$\rho_*(x) = \exp(2\Phi)$$

$$\rho(x,t) = \Psi(x,t)\, \rho_*^{1/2}(x)$$

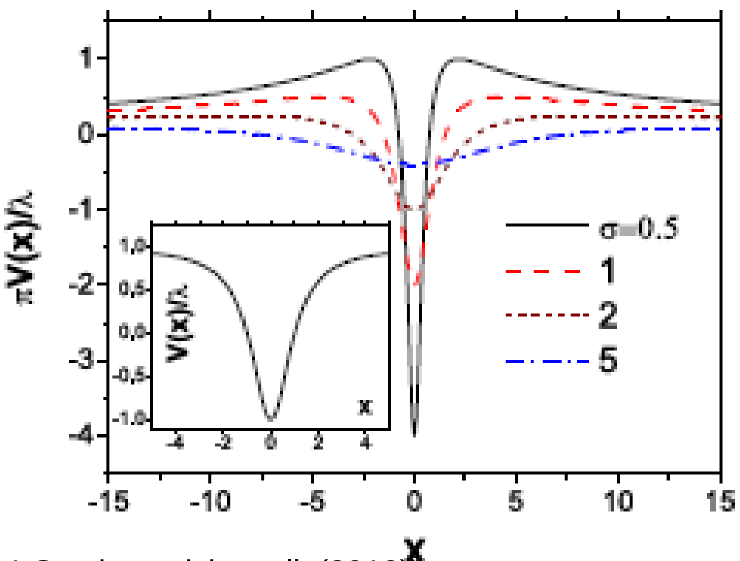
(Set λ=1 where necessary !)

$$\hat{H}_{\mu=1} \equiv |\nabla| + \mathcal{V}(x) \qquad \mathcal{V} = -\frac{|\Delta| \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\mathcal{V}(x) = \frac{\lambda}{\pi}\left[-\frac{2}{\sqrt{a}} + \frac{x}{a} \ln \frac{\sqrt{a}+x}{\sqrt{a}-x}\right]$$

$$a = \sigma^2 + x^2$$

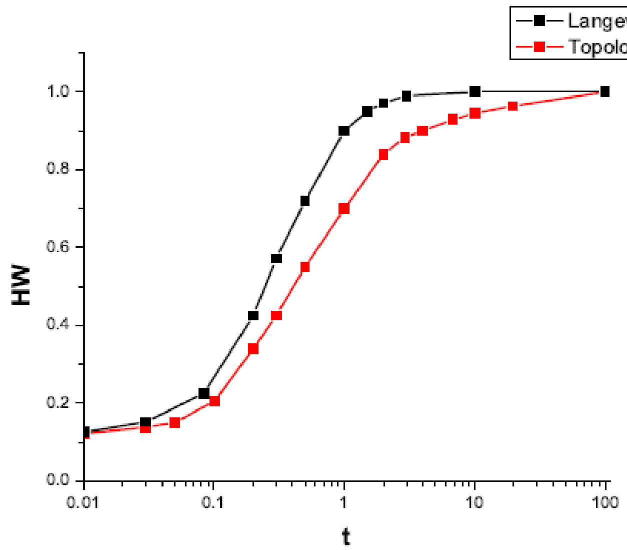
The potential is bounded from below and above –
„weak confinement", no pdf moments in existence.



(V. Stephanovich –coll. (2010))

Cauchy driver: targeted stochasticity in the time domain

Topological means **semigroup-induced**



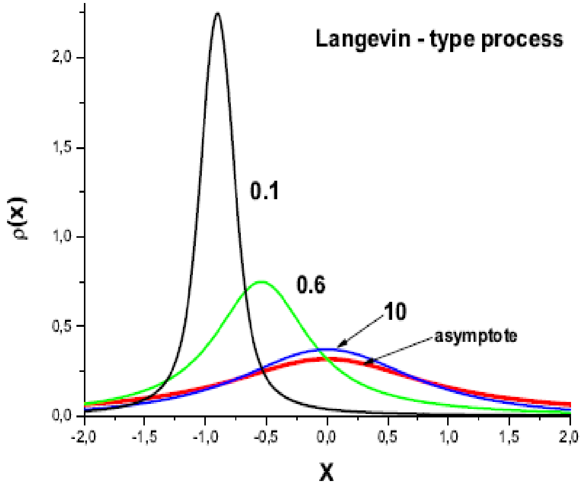


FIG. 1: Temporal behavior of the half-maximum width (HW): for the OUC process in Langevin-driven and semigroup-driven (topological) processes. Motions begin from common initial data $\rho(x,t=0)=\delta(x)$ and end up at a common pdf (20) for $\sigma=1$.

FIG. 2: Time evolution of Langevin-driven pdf $\rho_L(x,t)$ beginning from the initial data $\rho_L(x,t=0)=\delta(x+1)$ and ending at the pdf (20) (shown as "asymptote" in the figure) for $\sigma=1$. Figures near curves correspond to t values.

$$\rho(x,t)=\Psi(x,t)\,\rho_*^{1/2}(x)$$

OUC and semigroup -induced dynamics with the Cauchy target

$$\rho_*(x)=\frac{\sigma}{\pi(\sigma^2+x^2)}$$

Cauchy driver: targeted stochasticity in the time domain (confined noise)

Invariant density

$$\rho_*(x) = \frac{2}{\pi}\frac{1}{(1+x^2)^2}$$

Langevin drift

$$b(x) = -\frac{\gamma x}{8}(x^2+3)$$

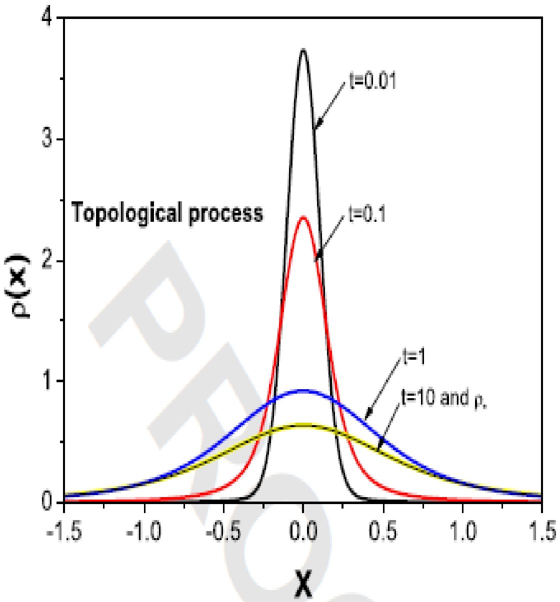
Semigroup potential

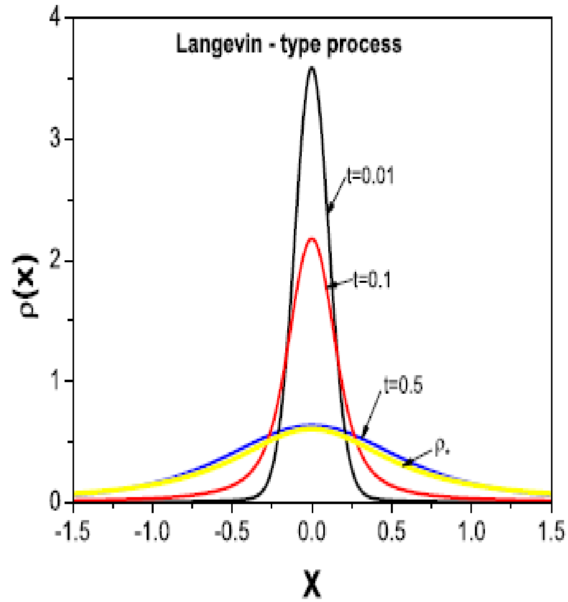
$$\mathcal{V}(x) := \lambda \frac{x^2-1}{x^2+1}$$

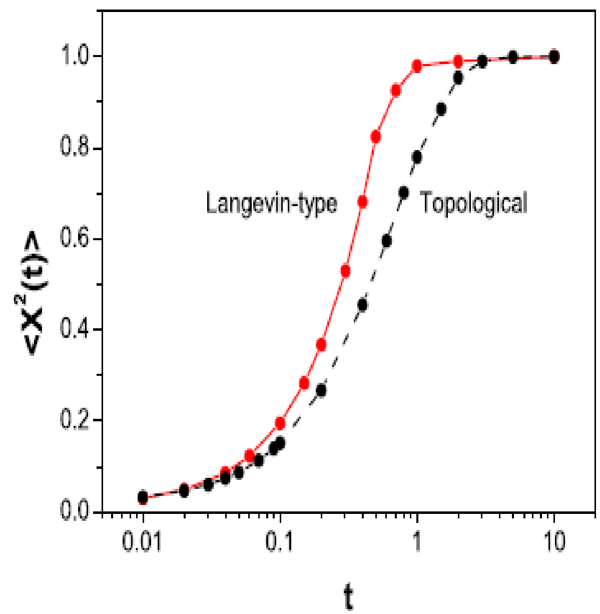
$$\Psi(x,t) = \rho(x,t)\, \rho_*^{-1/2}(x)$$

$$\hat{H}_\mu \equiv |\Delta| + \mathcal{V}(x)$$

Topological means **semigroup-induced**







Direct semigroup inference: **Cauchy oscillator** and the ground state –induced jump-type process

$$\hat{H}_{1/2} \equiv \lambda|\nabla| + \left(\frac{\kappa}{2}x^2 - \mathcal{V}_0\right)$$

$$\hat{H} = -D\Delta + \left(\frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2}\right)$$

direct reconstruction route:

$$\left(\frac{\kappa}{2}x^2 - \mathcal{V}_0\right)\rho_*^{1/2} = -\lambda\,|\nabla|\,\rho_*^{1/2}$$

$\tilde{f}(p)$ the Fourier transform of $f = \rho_*^{1/2}(x)$

$$-\frac{\kappa}{2}\Delta_p \tilde{f} + \gamma|p|\tilde{f} = \mathcal{V}_0 \tilde{f}$$

$k = (p - \sigma)/\zeta$ $\psi(k) = \tilde{f}(p)$ $\sigma = \mathcal{V}_0/\gamma$ $\zeta = (\kappa/2\gamma)^{1/3}$

$$\frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$$

Full spectral problem, c. f. Physica **A 389**, 4419, (2010) and Małecki and Lörinczi (2011)

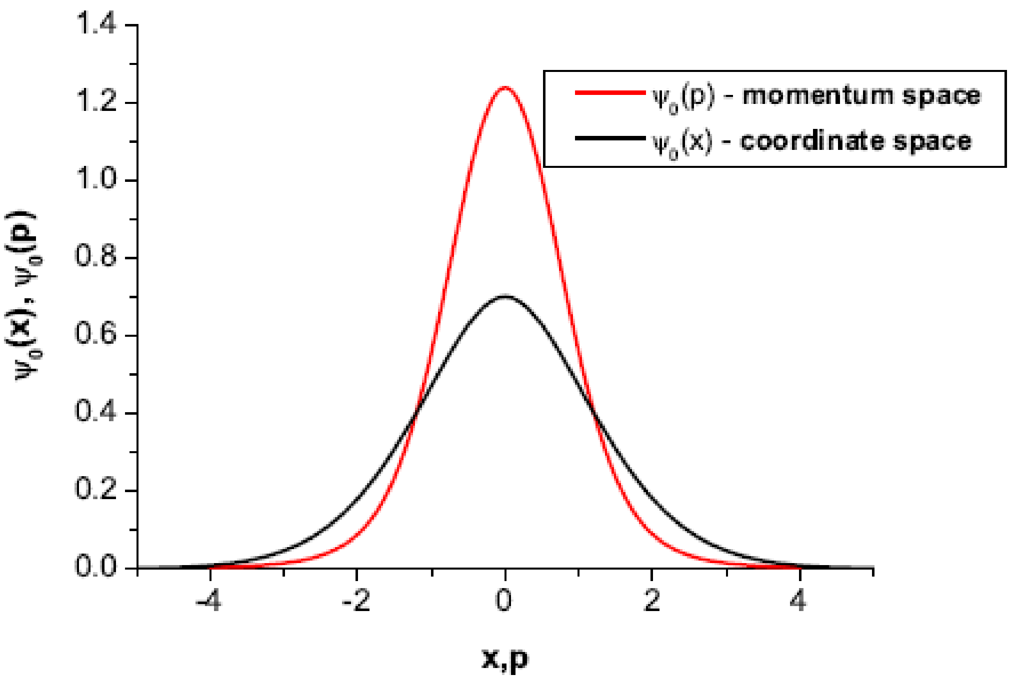
A unique normalized ground state function of

$$\frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$$

is composed of two Airy pieces that are glued together at the first zero y_0 of the Airy function derivative:

$$\psi_0(k) = A_0 \begin{cases} \mathrm{Ai}(-y_0 + k), & k > 0 \\ \mathrm{Ai}(-y_0 - k), & k < 0, \end{cases}$$

$$A_0 = \left[\mathrm{Ai}(-y_0)\sqrt{2y_0}\right]^{-1}, \quad y_0 \approx 1.01879297$$



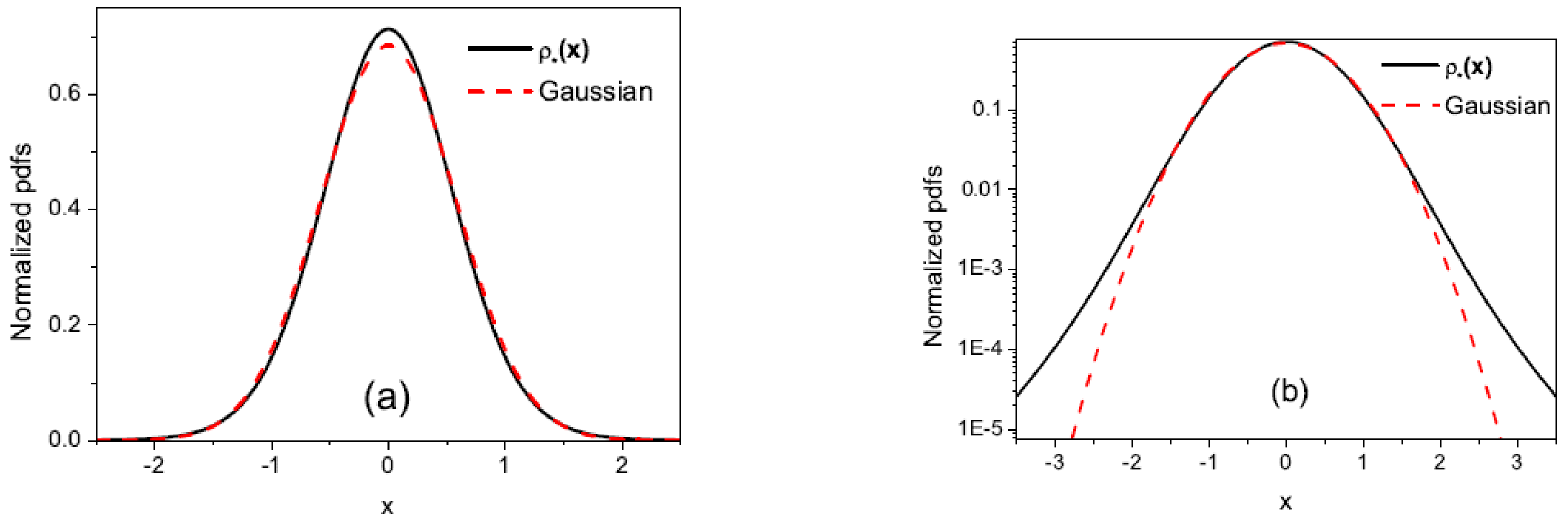


FIG. 7: Normalized invariant pdf (30) (full line) for the quadratic semigroup potential. The Gaussian function, centered at $x = 0$ and with the same variance $\sigma^2 = 0.339598$ is shown for comparison. Panel (a) shows functions in linear scale, while panel (b) shows them in logarithmic scale to better visualize their different behavior.

$$\psi_0(x) = \frac{A_0}{\pi} \int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t + y_0) dt = \rho_*^{1/2}(x)$$

K. Kaleta and T. Kulczycki (2009) **- asymptotic decay of eigenfunctions**

ψ_0 for $\hat{H}_\mu = |\Delta|^\mu + (x^2 - \mathcal{V}_0)$ displays $\sim |x|^{-(3+\mu)}$ heavy tail for $|x| \gg 1$

„Reverse engineering” for the Cauchy oscillator $\rho_* = \psi_0^2$ Langevin route

(Eliazar and Klafter (2003))

For a given ρ_* the definition of a drift function $b(x)$ (we put either $\lambda = 1$ or define $b \to b/\lambda$) is:

$$b(x) = -\frac{1}{\rho_*(x)} \int [|\nabla|\rho_*(x)]dx \equiv$$

$$\frac{1}{\pi\rho_*(x)} \int dx \int_{-\infty}^{\infty} \frac{\rho_*(x+y) - \rho_*(x)}{y^2} dy \,.$$

Inserting $\rho_*(x)$, Eq. (30), we get

Lévy- Langevin drift

$$b(x) = -\frac{\int_{-y_0}^{\infty} \mathrm{Ai}(t) \sin x(t+y_0)dt}{\int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t+y_0)dt} \,.$$

F-P equation

$$\partial_t \rho = -|\nabla|\rho - \nabla(b\,\rho)$$

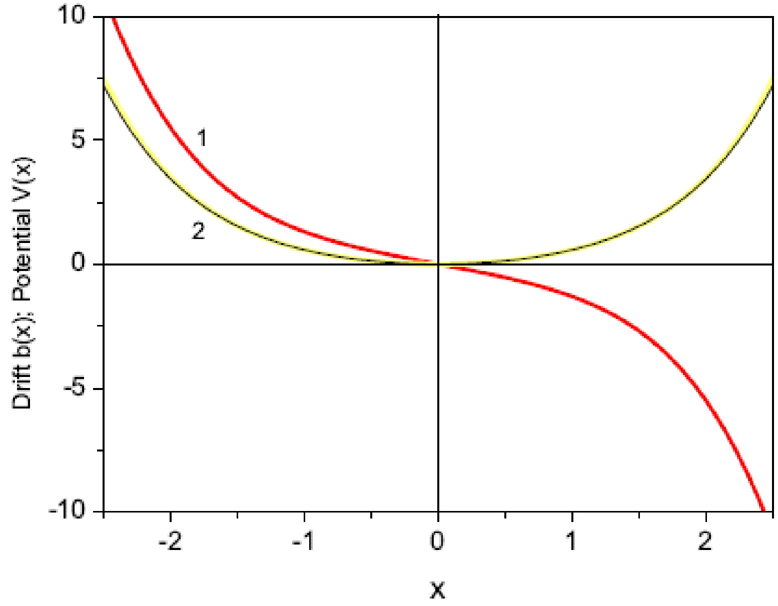


FIG. 8: Langevin - type drift $b(x)$ (curve 1) and its (force) potential $V(x)$ (curve 2), that give rise to an invariant density (30).

$$b(x) = -\nabla V(x), \quad V(x) \equiv - \int b(x) dx$$

$$\rho_* = \psi_0^2$$

$$\psi_0(x) = \frac{A_0}{\pi} \int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t + y_0) dt = \rho_*^{1/2}(x)$$

Confinement hierarchy - case study of a diffusion-type alternative

$$\rho_*(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha - 1/2)} \frac{1}{(1+x^2)^\alpha} \qquad \alpha > 1/2$$

Dynamical semigroup reconstruction

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\Psi(x,t) = \rho(x,t)\, \rho_*^{-1/2}(x)$$

(jump-type process)

Langevin drift reconstruction

$$b(x) = -\frac{\gamma}{\rho_*(x)} \int (|\nabla|\rho_*)(x)\, dx$$

$$\partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$\partial_t \rho_* = 0 = -\nabla(b\, \rho_*) - \gamma |\nabla| \rho_*$$

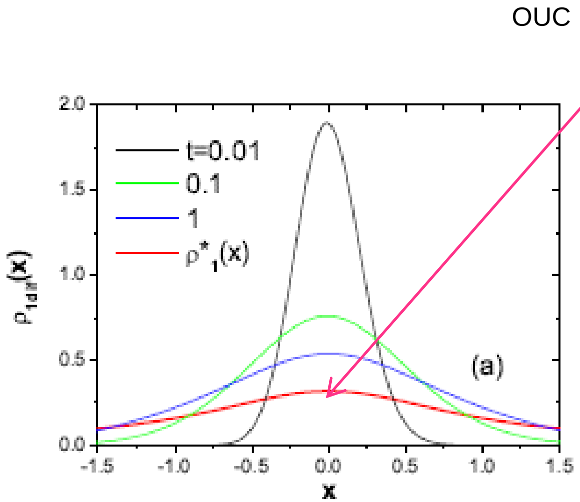
That was about **jump-type** processes. **What about diffusion-type alternative,** with the Gibbs-Boltzmann ansatz implicit, e.g.

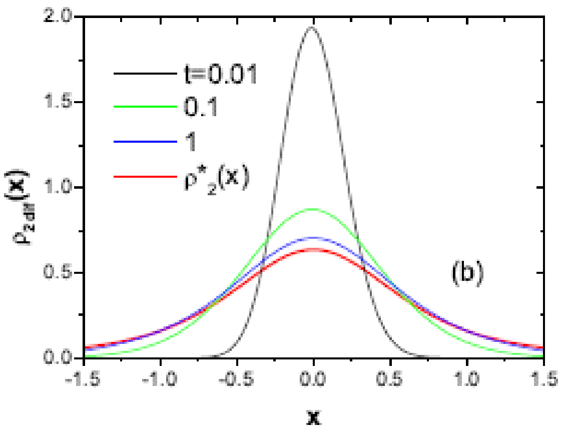
$$\rho_*(x) = C \exp(-\lambda V(x)).$$

$$\lambda = (k_B T)^{-1} \ (k_B \text{ is Boltzmann constant}).$$

Logarithmic potentials V(x) ~ $\ln(1+x^2)$ $\qquad b = -\nabla V(x) \sim -\nabla \ln \rho_*$

Abnormal (heavy-tailed) asymptotics of diffusion-type processes





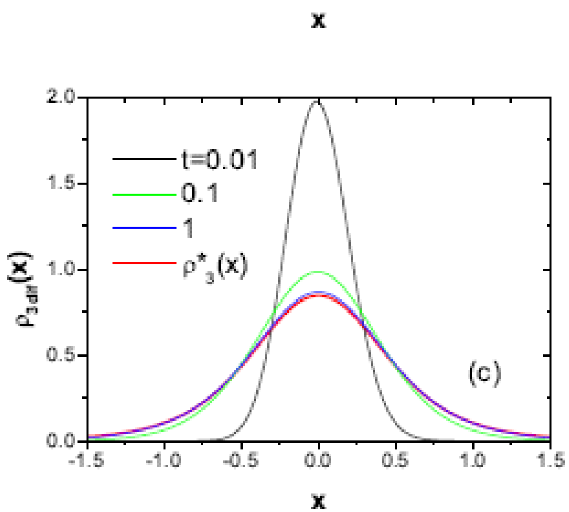


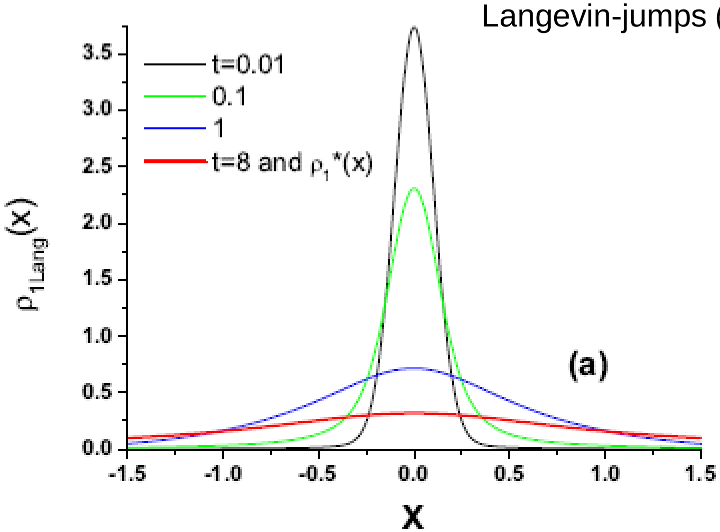
FIG. 6: Time evolution of pdf's $\rho(x,t)$ for Smoluchowski processes in logarithmic potential $\ln(1+x^2)$. The initial $(t=0)$ pdf is set to be a Gaussian with height 25 and half-width $\sim 10^{-3}$. The first depicted stage of evolution corresponds to $t = 0.01$. Target pdfs are the members of Cauchy family for $\alpha = 1, 2, 3$ respectively.

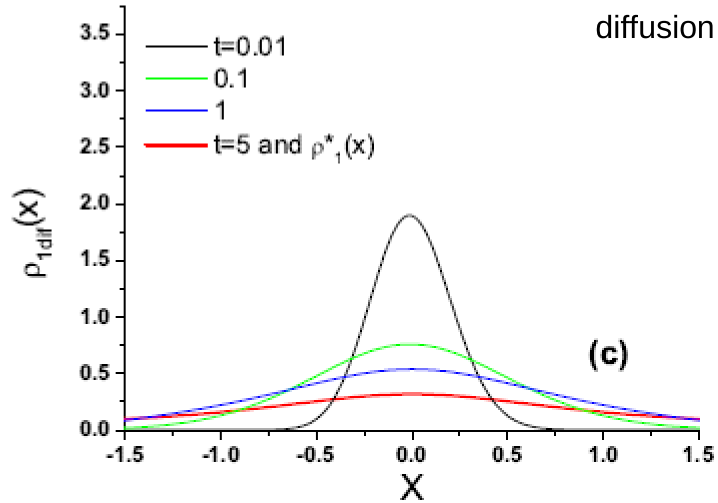
$$b = -\nabla V(x) \sim -\nabla \ln \rho_*$$

$$\rho_* \sim (1+x^2)^{-\alpha}$$

An equivalent semigroup dynamics does exist !

Langevin-jumps (OUC)





Diffusive scenario: Wiener driver,
Note ! Cauchy pdf is a target

Semigroup-jumps

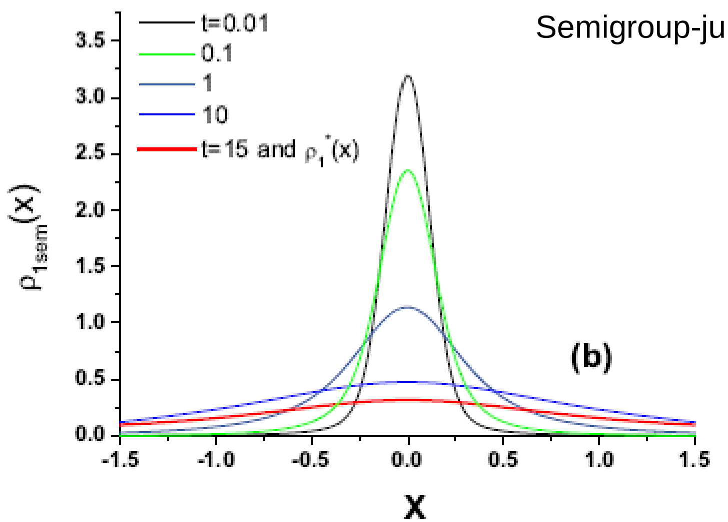


FIG. 2: Time evolution of pdf's $\rho(x,t)$ for the Cauchy-Langevin dynamics (panel (a)), Cauchy-semigroup-induced evolution (panel (b)) and the Wiener-Langevin process (panel (c)). The common target pdf is the Cauchy density, while the initial $t = 0$ pdf is set to be a Gaussian with height 25 and half-width $\sim 10^{-3}$. The first depicted stage of evolution corresponds to $t = 0.01$. The time rate hierarchy seems to be set: diffusion being fastest, next Lévy-Langevin and semigroup-driven evolutions being slower than previous two. However the outcome is not universal, as will show our further discussion.

$$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

Fractional quantum mechanics (rather dynamics **only**): „deeply condensed matter”

Notes:

(i) $L^2(R^n)$; $\hat{H}$ is a non-negative self-adjoint operator $\to -\hat{H}$ generates a strongly continuous positive semigroup $\exp(-t\hat{H})$

(ii) Take $\tau = t + is$, $Re\tau = t > 0$. Then $\exp(-\tau\hat{H})$ is holomorphic. (iii) Consider $t \downarrow 0$, we are left with a unitary group $\exp(-is\hat{H})$. Re-define $s \to t \in R$.

We have $i\partial_t\psi = \hat{H}\psi$ and $\rho(x,t) = (\psi\overline{\psi})(x,t)$

Use $\hat{H}_\mu$ or a quasi-relativistic Hamiltonian, we get a pseudo-differential (fractional included) quantum dynamics. How does $\rho(x,t)$ behave ?

$$i\partial_t\Psi = \hat{H}_\mu\Psi$$

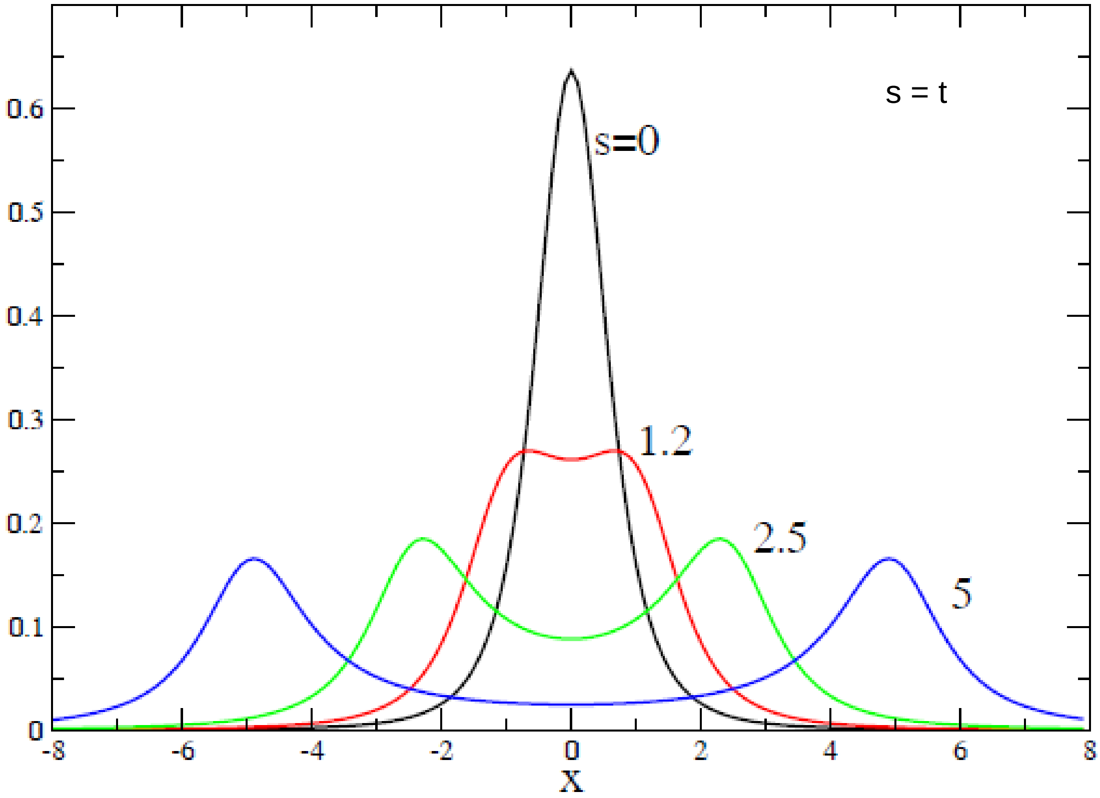
$$\hat{H}_\mu \equiv |\Delta|^{\mu/2} + \mathcal{V}(x)$$

If $\rho_*^{1/2}$ is a ground state with the eigenvalue 0, then clearly $\mathcal{V} = -\frac{|\Delta|^{\mu/2}\rho_*^{1/2}}{\rho_*^{1/2}}$

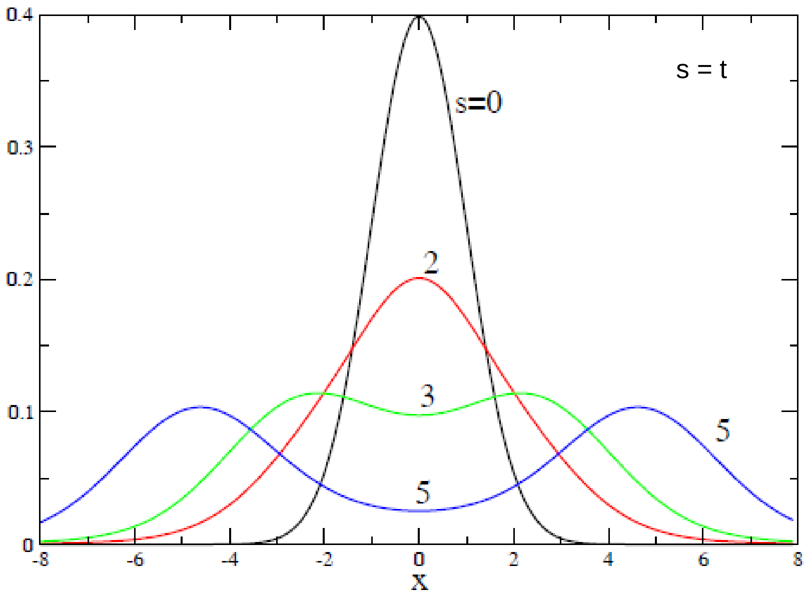
Free fractional evolution with the 1D Cauchy driver

$$i\partial_t \psi = |\nabla|\psi$$

$$\psi(x,0) = \sqrt{\frac{2}{\pi}}\frac{1}{1+x^2} \Longrightarrow |\psi|^2(x,t) = \frac{2}{\pi}\frac{1+t^2}{[1+(x-t)^2][1+(x+t)^2]}$$



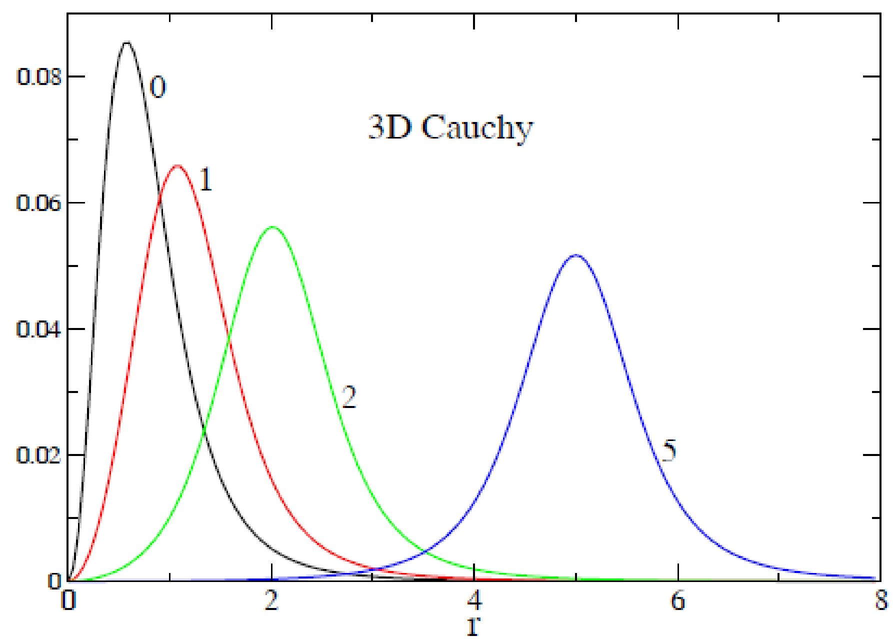
Enhanced delocalization: dynamically developed bimodality of the pdf



$$\psi(x,0) = \frac{1}{\sqrt[4]{2\pi}} e^{-x^2/4}$$

$$\psi_{Gauss}(x,s) = \frac{1}{2\sqrt[4]{2\pi}} \left\{ \left[e^{-\frac{(x-s)^2}{4}} + e^{-\frac{(x+s)^2}{4}} \right] + \frac{i}{2} P \left[\int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-s)^2}{4}}}{y} dy - \frac{e^{-\frac{(x-y+s)^2}{4}}}{y} dy \right] \right\}$$

Radial dependence of the pdf for **3D Cauchy driver**



Pdfs dynamical decay through
decaying heavy tails

Instead of a summary: some (minor ?) quieries/problems

Lévy flights - well explored, sample path properties and Markovianess isues are amenable to analytic and numerical methods

Lévy-Schrödinger semigroups – the induced pdf dynamics needs further exploration, no rigorous results about X(t), sample path properties basically unknown (work in progress with a numerical assistance) , Markovianess not necessarily obvious, albeit seemingly valid

Heavy-tailed asymptotics of diffusion-type processes – not obvious whether valid for arbitrary (not generic !) initial data, possible (unexplored) problems with known lower/upper bounds for solutions of parabolic equations e.g. text-book „properties of solutions of the Fokker-Planck equation". Support coming from the semigroup reformulation

Pseudo-differential QM – spectral problems are hard, only a limited number of solvable cases (stability of matter issues are well developed, not mentioned here). Scarse analytic form of ground states The induced pdf dynamics is realized in terms of jumps, with well defined jump kernels. No reliable results on sample paths behavior. Markovianess under scrutiny. The ground-state process limited to an invariant measure (normalized square of the lowest positive eigenfunction), of no physical relevance beyond this regime.