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Stochastic Mechanics and the Kepler Problem: Scattering States

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Summary

We establish a manifest connection between the classical and quantized versions of the Kepler problem in case of scattering states. The affinity to the four-oscillator system allows to deduce the stochastic description of the Kepler problem, which in the leading order of the tree (semi-classical) approximation is given in terms of three independent Markov processes the latter being driven by the classical Kepler motion.

1. Motivation

In the previous paper [1] a possibility has been examined of incorporating the quantum Kepler problem in the framework of the stochastic mechanics [2—4]. The discrete (bound state) spectrum version of the problem was considered and its affinity to the four-oscillator model made explicit, both on the quantum and classical levels of theory. The classical-quantum relationship was recovered through computing coherent state expectation values of operator expressions: the quantized Kepler problem was represented in the Hilbert space of the four-oscillator system. The case of scattering states was left aside in [1] and it is the main purpose of the present investigation to associate stochastic processes to continuous spectrum variant of the quantum Kepler problem.

To exemplify the stochastic mechanics strategy [2, 3], les us consider a quantum mechanical problem, where

$$H = -\frac{\hbar^2}{2m} \Delta + V \tag{1.1}$$

is the Hamiltonian operator acting on the wave function $\psi = \psi(x, t)$:

$$\psi = \varrho^{1/2} \exp (iS/\hbar) . \tag{1.2}$$

Then [4]

$$\int \bar{\psi}(x) (H\psi) (x) \, dx = \mathcal{H}(\varrho, S) = \int \left(\frac{1}{2} mv^2 + \frac{1}{2} mu^2 + V \right) (x) \, \varrho(x) \, dx \tag{1.3}$$

where

$$v = v(x, t) = \nabla S/m$$
$$u = u(x, t) = \left(\frac{\hbar}{2m} \right) \frac{\nabla \varrho}{\varrho} \tag{1.4}$$

and the Hamilton-Jacobi-Mandelung equation

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla(\varrho^{1/2})}{\varrho^{1/2}} = 0 \tag{1.5}$$

holds true together with

$$\partial_t = -\nabla(\varrho v). \tag{1.5a}$$

According to [4] ϱ and S can be viewed as the phase-space variables with the Poisson bracket

$$\{\mathcal{A}, \mathcal{B}\} = \int \left[\frac{\delta \mathcal{A}}{\delta \varrho(x)} \frac{\delta \mathcal{B}}{\delta S(x)} - \frac{\delta \mathcal{A}}{\delta S(x)} \frac{\delta \mathcal{B}}{\delta \varrho(x)}\right] dx \tag{1.6}$$

so that

$$\begin{aligned} &\{\varrho(x), S(x')\} = \delta(x - x') \\ &\partial_t \varrho = \{\varrho(x, t), \mathcal{H}\} = \frac{\delta \mathcal{H}}{\delta S(x, t)} \\ &\partial_t S = \{S(x, t), \mathcal{H}\} = -\frac{\delta \mathcal{H}}{\delta \varrho(x, t)}. \end{aligned} \tag{1.7}$$

Since, with respect to (1.6), there holds:

$$\begin{aligned} &\{\psi(x), \overline{\psi}(x')\} = \delta(x - x') \frac{1}{i\hbar} \\ &\{\psi(x), \psi(x')\} = 0 = \{\overline{\psi}(x), \overline{\psi}(x')\} \end{aligned} \tag{1.8}$$

we arrive at conclusion that

$$\partial_t \psi = \{\psi, \mathcal{H}\} = \frac{1}{i\hbar} \frac{\delta \mathcal{H}}{\delta \overline{\psi}(x)} = \frac{1}{i\hbar} (H\psi)(x) \tag{1.9}$$

i.e. $\psi = \psi(x, t)$ solves the Schrödinger equation.

Furthermore, for all observables which can be written in the form

$$\begin{aligned} &(A\psi)(x) = \int A(x, x') \psi(x') dx' \\ &\mathcal{A}(\varrho, S) = \iint \overline{\psi}(x) A(x, x') \psi(x') dx dx' \end{aligned} \tag{1.10}$$

there holds

$$\{\mathcal{A}, \mathcal{B}\} = \frac{1}{i\hbar} \int dx \overline{\psi}(x) \{[A, B]_- \psi\}(x) \tag{1.11}$$

where $\mathcal{A} = \mathcal{A}(\varrho, S)$.

If to specialize this discussion to the harmonic oscillator, the stochastic background of the model is best seen if ψ's are the coherent states [3]:

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + \frac{1}{2} m\omega^2 x^2 \psi \tag{1.12}$$

$$\psi(x,t) = (2\pi\sigma)^{-1/4} \exp\left[-\frac{1}{4\sigma}\left(x - q_{cl}(t)\right)^2 + \frac{i}{\hbar} x p_{cl}(t) - \frac{i}{2\hbar} p_{cl}(t)\, q_{cl}(t) - i\frac{\omega}{2} t\right]$$
$$\sigma = \frac{\hbar}{2m\omega} \tag{1.12}$$

since then

$$\varrho(x,t) = (2\pi\sigma)^{-1/2} \exp\left[-\frac{1}{2\sigma}\left(x - q_{cl}(t)\right)^2\right]$$
$$S(x,t) = x p_{cl}(t) - \frac{1}{2} p_{cl}(t)\, q_{cl}(t) - \frac{1}{2}\hbar\omega t \tag{1.13}$$

and the coherent state expectation value of the configuration operator $\hat{q}$ equals $q_{cl}(t)$ which together with $p_{cl}(t)$ determine the classical harmonic motion, while $q_{cl} = \int dx\, x \varrho(x)$, $p_{cl} = \int \varrho(x)\, \nabla S(x)\, dx$. The generalization of this harmonic oscillator discussion to the four-oscillator case is apparent and was exploited in [1] for the construction of the stochastic mechanics of the (discrete spectrum) quantized Kepler problem in terms of the four-oscillator stochastic processes which are subject to constraints. By stochastic mechanics we mean the validity of formulas of the type (1.6), (1.7) and for the Kepler problem the constrained classical four-oscillator motion determines the appropriate densities and phases of the stochastic process. This point is discussed in more detail in Section 4 of the present paper.

2. Passage to the Classical Kepler Problem in Case of the Continuous Spectrum

Let us consider the reduced Hamiltonian of the Coulomb problem

$$H = -\frac{\hbar^2}{2m} \nabla^2 - \frac{Ze^2}{r}. \tag{2.1}$$

The Laplace-Runge-Pauli operator $\boldsymbol{M}$ obeys

$$\boldsymbol{M}^2 - (Ze^2)^2 = \frac{2H}{m} (\boldsymbol{L}^2 + \hbar^2) \tag{2.2}$$

and we know that

$$[\boldsymbol{L}^2, \boldsymbol{M}^2]_- = 0 = [H, \boldsymbol{M}]_-$$
$$\boldsymbol{L} \cdot \boldsymbol{M} = \boldsymbol{M} \cdot \boldsymbol{L}. \tag{2.3}$$

In the case of the continuous spectral problem for (2.1) following Kibler and Negadi [5, 6] we define (notice that the operator H appears in the denominator)

$$\boldsymbol{B} = \left(\frac{m}{2H}\right)^{1/2} \boldsymbol{M} \tag{2.4}$$

so that equations (2.2) and (2.3) imply

$$\boldsymbol{L}^2 + \boldsymbol{B}^2 + \hbar^2 = -\left(\frac{m}{2H}\right)(Ze^2)^2 \tag{2.5}$$

$$\boldsymbol{L} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{L}. \tag{2.6}$$

The number of six operators (L_1, L_2, L_3) and (B_1, B_2, B_3) constitutes the Lie algebra of the Lorentz group $SO(3, 1)$

$$\begin{aligned} &[L_j, L_k]_- = i\hbar\varepsilon_{jkl}L_l, \qquad [L_j, B_k]_- = i\hbar\varepsilon_{jkl}B_l \\ &[B_j, B_k]_- = -i\hbar\varepsilon_{jkl}L_l. \end{aligned} \tag{2.7}$$

We shall introduce a particular (Jordan's) bosonic representation of (2.7) given by

$$\begin{aligned} L_j &= \frac{1}{2}(a^+\sigma_j a + b^+\sigma_j b)\,\hbar \\ & \qquad\qquad\qquad\qquad j = 1, 2, 3 \\ B_j &= \frac{1}{2}(a^+\sigma_j C b^{+T} - a^T C\sigma_j b)\,\hbar \end{aligned} \tag{2.8}$$

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \qquad b = \begin{pmatrix} a_3 \\ a_4 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

σ_j — Pauli matrices

where the Fock representation of the CCR algebra with generators

$$[a_k, a_j^+]_- = \delta_{kj}, \qquad [a_k, a_j]_- = [a_k^+, a_j^+] = 0 \tag{2.9}$$

where $j, k = 1, 2, 3, 4$, $a_k\,|0\rangle = 0$, is in use.

Inserting (2.8) into (2.5) and (2.6) we arrive at the following operator identities

$$\begin{aligned} &(a_1^+a_1 + a_2^+a_2 - a_3^+a_3 - a_4^+a_4)^2 - (a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2 \\ &= -\frac{2m}{H}\frac{(Ze^2)^2}{\hbar^2} \end{aligned} \tag{2.10}$$

$$(a_1^+a_1 + a_2^+a_2 - a_3^+a_3 - a_4^+a_4)\,(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3) = 0. \tag{2.11}$$

Since the two factors in (2.11) commute and the continuous spectrum arises if $E > 0$, we finally get

$$a_1^+a_1 + a_2^+a_2 - a_3^+a_3 - a_4^+a_4 = 0 \tag{2.12}$$

$$H = \frac{2m(Ze^2)^2}{\hbar^2}\,\frac{1}{(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2} \tag{2.13}$$

which is the scattering variant of the Kepler Hamiltonian in the boson realization: the constraint equation (2.12) should be accounted for while solving the spectral problem for H.

Let us introduce the new operators:

$$\begin{aligned} \hat{Q}_j &= \frac{1}{2}\left(\frac{2\hbar}{m\omega}\right)^{1/2}(a_j + a_j^+) \\ \hat{P}_j &= \frac{1}{2i}(2\hbar m\omega)^{1/2}(a_j - a_j^+), \qquad j = 1, 2, 3, 4, \end{aligned} \tag{2.14}$$

in terms of which (2.12), (2.13) reads

$$\frac{1}{\hbar\omega}\left[\frac{m\omega^2}{2}(\hat{Q}_1^2+\hat{Q}_2^2-\hat{Q}_3^2-\hat{Q}_4^2)+\frac{1}{2m}(\hat{P}_1^2+\hat{P}_2^2-\hat{P}_3^2-\hat{P}_4^2)\right]=0 \tag{2.15}$$

$$H=\frac{m\omega^2(Ze^2)^2}{2}\frac{1}{\left[\frac{m\omega^2}{2}(\hat{Q}_1\hat{Q}_4-\hat{Q}_2\hat{Q}_3)-\frac{1}{2m}(\hat{P}_1\hat{P}_4-\hat{P}_2\hat{P}_3)\right]^2}. \tag{2.16}$$

To pass to the classical version of this quantum problem, we shall adopt the coherent state technique, which was previously used in the discrete (bound states) spectral variant of the Kepler problem [1, 7]. The classical phase-space variables and the classical Hamiltonian are supposed to arise via taking the coherent state expectation values of operator quantities in the tree approximation.

$$Q_j := \langle\alpha|:\hat{Q}_j:|\alpha\rangle = \langle\alpha|\,\hat{Q}_j\,|\alpha\rangle = \frac{1}{2}\left(\frac{2\hbar}{m\omega}\right)^{1/2}(\alpha_j+\bar{\alpha}_j) \tag{2.17a}$$

$$P_j := \langle\alpha|:\hat{P}_j:|\alpha\rangle = \langle\alpha|\,\hat{P}_j\,|\alpha\rangle = \frac{1}{2i}(2\hbar m\omega)^{1/2}(\alpha_j-\bar{\alpha}_j)$$

$$H^c = \langle\alpha|:H:|\alpha\rangle, \qquad j=1,2,3,4. \tag{2.17}$$

Here, the tree approximation amounts to the normal ordering $:\,.\,:$ prescription for all operators, while $|\alpha\rangle$ is the coherent four-oscillator state

$$|\alpha\rangle = \exp\left(-\frac{1}{2}\sum_{i=1}^{4}|\alpha_i|^2\right)\exp\left(\sum_{i=1}^{4}\alpha_i a_i^+\right)|0\rangle \tag{2.18a}$$

$$a_i\,|0\rangle = 0 \qquad i=1,2,3,4 \tag{2.18b}$$

$$a_i\,|\alpha\rangle = \alpha_i\,|\alpha\rangle. \tag{2.18c}$$

H is to be viewed as a formal power series

$$\begin{aligned}H &= \frac{2m(Ze^2)^2}{\hbar^2}\frac{1}{1+[(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2-1]}\\ &= \frac{2m(Ze^2)^2}{\hbar^2}\big(1-[(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2-1]\\ &\quad + [(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2-1]^2\\ &\quad - [(a_1^+a_4^+ + a_1a_4 - a_2^+a_3^+ - a_2a_3)^2-1]^3+\dots\big)\end{aligned} \tag{2.19}$$

so that the normal ordering can be easily effected. Thus

$$\begin{aligned}H^c &= \frac{2m(Ze^2)^2}{\hbar^2}\frac{1}{(\bar{\alpha}_1\bar{\alpha}_4+\alpha_1\alpha_4-\bar{\alpha}_2\bar{\alpha}_3-\alpha_2\alpha_3)^2}\\ &= \frac{m(Ze^2)^2\,\omega^2}{2}\frac{1}{\left[\frac{m\omega^2}{2}(Q_1Q_4-Q_2Q_3)-\frac{1}{2m}(P_1P_4+P_2P_3)\right]^2}\end{aligned} \tag{2.20}$$

and additionally, from (2.15) we get

$$\frac{m\omega^2}{2}(Q_1^2+Q_2^2-Q_3^2-Q_4^2)+\frac{1}{2m}(P_1^2+P_2^2-P_3^2-P_4^2)=0. \tag{2.21}$$

After the canonical transformation

$$\begin{aligned}
P_1' &= \frac{1}{\sqrt{2}}(P_1+P_4) & Q_1' &= \frac{1}{\sqrt{2}}(Q_1+Q_4)\\
P_2' &= \frac{1}{\sqrt{2}}(P_2-P_3) & Q_2' &= \frac{1}{\sqrt{2}}(Q_2-Q_3)\\
P_3' &= \frac{m\omega}{\sqrt{2}}(Q_2+Q_3) & Q_3' &= -\frac{1}{\sqrt{2}}\frac{1}{m\omega}(P_2+P_3)\\
P_4' &= \frac{m\omega}{\sqrt{2}}(Q_1-Q_4) & Q_4' &= \frac{1}{\sqrt{2}}\frac{1}{m\omega}(P_4-P_1)
\end{aligned} \tag{2.22a}$$

$$\begin{aligned}
&\{P_i', P_j'\}_{Q,P} = \{Q_i', Q_j'\}_{Q,P} = 0\\
&\{Q_i', P_j'\}_{Q,P} = \delta_{ij}, \qquad i,j = 1,2,3,4,
\end{aligned} \tag{2.22b}$$

($\{,\}_{Q,P}$ means the Poisson bracket) we pass to the following expression for the Hamiltonian (2.20)

$$H^c = 2m\omega^2(Ze^2)^2\frac{1}{\left(\frac{1}{2m}\sum_{i=1}^{4}P_i'^2-\frac{1}{2}m\omega^2\sum_{i=1}^{4}Q_i'^2\right)^2}. \tag{2.23}$$

The constraint (2.21) becomes transformed to the form

$$Q_1'P_4' - Q_4'P_1' + Q_2'P_3' - Q_3'P_2' = 0. \tag{2.24}$$

(2.23) does not resemble the classical Kepler Hamiltonian (unlike to the discrete spectrum problem of Ref. [1]). However at this point we can apply the Kustaanheimo-Stiefel transformation. We introduce [6] the matrix (we neglect primes)

$$A = \begin{pmatrix} Q_3 & -Q_4 & Q_1 & -Q_2\\ Q_2 & Q_1 & Q_4 & Q_3\\ -Q_1 & Q_2 & Q_3 & -Q_4\\ -Q_4 & -Q_3 & Q_2 & Q_1 \end{pmatrix} \tag{2.25}$$

and the new phase-space variables (q_1, q_2, q_3), (p_1, p_2, p_3) as

$$\begin{pmatrix} q_1\\ q_2\\ q_3\\ 0\end{pmatrix} = A\begin{pmatrix} Q_1\\ Q_2\\ Q_3\\ Q_4\end{pmatrix} \tag{2.26a}$$

$$\begin{pmatrix} p_1\\ p_2\\ p_3\\ 0\end{pmatrix} = \frac{1}{2r}A\begin{pmatrix} P_1\\ P_2\\ P_3\\ P_4\end{pmatrix}, \tag{2.26b}$$

where

$$r = Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = (q_1^2 + q_2^2 + q_3^2)^{1/2}. \tag{2.26c}$$

The new variables are the canonical ones (see Appendix A), because of

$$\begin{aligned} &\{q_i, q_j\}_{Q,P} = 0 \\ &\{p_i, p_j\}_{Q,P} \approx 0 \qquad i, j = 1, 2, 3, 4 \\ &\{q_i, p_j\}_{Q,P} = \delta_{ij}. \end{aligned} \tag{2.27}$$

Here the second bracket is equated to zero on the surface of constraints (therefore we use $\approx$ instead of the equality sign $=$). Upon admitting that the classical energy $E = H^C = 1/8m\omega^2$, the Hamiltonian (2.23) acquires now the form:

$$H^C = \frac{1}{2m} \boldsymbol{P}^2 - \frac{Ze^2}{r}. \tag{2.28}$$

Hence we arrive at the classical Kepler Hamiltonian.

3. Passage to the Classical Equation of Motion

In case of the discrete (bound state) spectrum, the Kepler problem was represented in the Hilbert space of the (attractive) harmonic oscillator. Presently, instead of bound states, the scattering states are of interest for us. The affinity to the harmonic oscillator can be recovered now as well, but the repulsive potential should replace the previous attractive one. Let us denote

$$H_0' = \hbar\omega \sum_{i=1}^{4} \left(a_i^+ a_i + \frac{1}{2}\right) \tag{3.1a}$$

$$H' = -\frac{m(Ze^2)^2}{\hbar^2} \frac{1}{(a_1^+ a_1 + a_2^+ a_2 + 1)^2 + (a_3^+ a_3 + a_4^+ a_4 + 1)^2} \tag{3.1b}$$

$$H_0 = \frac{1}{2} \hbar\omega \sum_{i=1}^{4} (a_i a_i + a_i^+ a_i^+) \qquad \text{(Appendix C)} \tag{3.2a}$$

$$H = \frac{2m(Ze^2)^2}{\hbar^2} \frac{1}{(a_1^+ a_4^+ + a_1 a_4 - a_2^+ a_3^+ - a_2 a_3)^2} \tag{3.2b}$$

where H_0 is the Hamiltonian for the repulsive harmonic problem. We have

$$[H_0, H]_- = 0, \qquad [H_0', H']_- = 0 \tag{3.3}$$

but the H_0, H pair does not commute with the H_0', H' one.

In quantum theory, the Heisenberg equation

$$\frac{d\hat{F}_H}{dt} = \frac{i}{\hbar} [H, \hat{F}_H]_- \tag{3.4}$$

provides us with the time development of operators. After accounting for (see Appendix B)

$$[H, a_1]_- = \frac{1}{Y} a_4^+ H + \frac{1}{Y^2} a_4^+ YH$$

$$[H, a_2]_- = -\frac{1}{Y} a_3^+ H - \frac{1}{Y^2} a_3^+ YH$$

$$[H, a_3]_- = -\frac{1}{Y} a_2^+ H - \frac{1}{Y^2} a_2^+ YH \tag{3.5a}$$

$$[H, a_4]_- = \frac{1}{Y} a_1^+ H + \frac{1}{Y^2} a_1^+ YH$$

$$[H, a_1^+]_- = -\frac{1}{Y} a_4 H - \frac{1}{Y^2} a_4 YH$$

$$[H, a_2^+]_- = \frac{1}{Y} a_3 H + \frac{1}{Y^2} a_3 YH$$

$$[H, a_3^+]_- = \frac{1}{Y} a_2 H + \frac{1}{Y^2} a_2 YH \tag{3.5b}$$

$$[H, a_4^+]_- = -\frac{1}{Y} a_1 H - \frac{1}{Y^2} a_1 YH$$

where

$$Y = a_1^+ a_4^+ + a_1 a_4 - a_2^+ a_3^+ - a_2 a_3 \tag{3.5c}$$

and denoting by F the classical analogue of the operator $\hat{F}$

$$F = \langle \alpha | :\hat{F}: | \alpha \rangle \tag{3.6}$$

we arrive at the following equations of motion (see Appendix B)

$$\dot{Q}_k = \langle \alpha | :\frac{d\hat{Q}_k}{dt}: | \alpha \rangle = \frac{i}{\hbar} \langle \alpha | :[H, \hat{Q}_k]: | \alpha \rangle = \frac{\partial H^c}{\partial P_k}$$

$$\dot{P}_k = \langle \alpha | :\frac{d\hat{P}_k}{dt}: | \alpha \rangle = \frac{i}{\hbar} \langle \alpha | :[H, \hat{P}_k]_-: | \alpha \rangle = -\frac{\partial H^c}{\partial Q_k}, \qquad k = 1, 2, 3, 4 \tag{3.7a}$$

$$H^c = \frac{m\omega^2 (Ze^2)^2}{2} \frac{1}{\left[\frac{m\omega^2}{2} (Q_1 Q_4 - Q_2 Q_3) - \frac{1}{2m} (P_1 P_4 - P_2 P_3)\right]^2} \tag{3.7b}$$

where $:[H, \hat{Q}]_-:$ means first computing the commutator and next taking the normal ordered form of the resulting expression. Analogously we can proceed in case of the discrete spectrum (it supplements our previous consideration [1]):

$$[H', a_1]_- = \frac{1}{X} (a_1^+ a_1 + a_2^+ a_2 + 1)\, a_1 H' + \frac{1}{X} a_1 (a_1^+ a_1 + a_2^+ a_2 + 1)\, H'$$

$$[H', a_2]_- = \frac{1}{X} (a_1^+ a_1 + a_2^+ a_2 + 1)\, a_2 H' + \frac{1}{X} a_2 (a_1^+ a_1 + a_2^+ a_2 + 1)\, H' \tag{3.8a}$$

$$[H', a_3]_- = \frac{1}{X}(a_3^+a_3 + a_4^+a_4 + 1)\,a_3H' + \frac{1}{X}\,a_3(a_3^+a_3 + a_4^+a_4 + 1)\,H'$$

$$[H', a_4]_- = \frac{1}{X}(a_3^+a_3 + a_4^+a_4 + 1)\,a_4H' + \frac{1}{X}\,a_4(a_3^+a_3 + a_4^+a_4 + 1)\,H' \tag{3.8a}$$

$$[H', a_1^+]_- = -\frac{1}{X}(a_1^+a_1 + a_2^+a_2 + 1)\,a_1^+H' - \frac{1}{X}\,a_1^+(a_1^+a_1 + a_2^+a_2 + 1)\,H'$$

$$[H', a_2^+]_- = -\frac{1}{X}(a_1^+a_1 + a_2^+a_2 + 1)\,a_2^+H' - \frac{1}{X}\,a_2^+(a_1^+a_1 + a_2^+a_2 + 1)\,H' \tag{3.8b}$$

$$[H', a_3^+]_- = -\frac{1}{X}(a_3^+a_3 + a_4^+a_4 + 1)\,a_3^+H' - \frac{1}{X}\,a_3^+(a_3^+a_3 + a_4^+a_4 + 1)\,H'$$

$$[H', a_4^+]_- = -\frac{1}{X}(a_3^+a_3 + a_4^+a_4 + 1)\,a_4^+H' - \frac{1}{X}\,a_4^+(a_3^+a_3 + a_4^+a_4 + 1)\,H'$$

where

$$X = (a_1^+a_1 + a_2^+a_2 + 1)^2 + (a_3^+a_3 + a_4^+a_4 + 1)^2 \tag{3.8c}$$

so that after computing the coherent state expectation values in the tree approximation, we obtain (Appendix B):

$$\dot{Q}_k = \frac{\partial H'^c}{\partial P_k}, \qquad \dot{P}_k = -\frac{\partial H'^c}{\partial Q_k}, \qquad k = 1, 2, 3, 4 \tag{3.9a}$$

$$H'^c = \langle\alpha|:H':|\alpha\rangle = -m\omega^2(Ze^2)^2 \times \frac{1}{\left[\frac{m\omega^2}{2}(Q_1^2 + Q_2^2) + \frac{1}{2m}(P_1^2 + P_2^2)\right]^2 + \left[\frac{m\omega^2}{2}(Q_3^2 + Q_4^2) + \frac{1}{2m}(P_3^2 + P_4^2)\right]^2}. \tag{3.9b}$$

H'^c is the classical Hamiltonian for the Kepler problem with $E < 0$ (see [1]), (3.7) from which after taking the canonical transformation (2.22) we get (3.9b).

Remark

In the case of H_0 or H_0' we do not need the normal ordering (tree approximation) because of the simplicity of the model. The coherent state averages of Heisenberg equations coincide with the Hamilton equations for classical variables (Appendix C).

4. Stochastic Mechanics of the Continuum Kepler Problem

We would like to construct a stochastic scheme for the Kepler problem following the standard harmonic oscillator route (see [3]). In our case there is one difficulty. So far we have analysed the classical-quantum relationship for the Kepler problem, where Heisenberg operators are represented in the Hilbert space of the four-oscillator system. Hence no time dependence of state vectors was involved. However, we must pass to the Schrödinger picture, to fit with the stochastic mechanics idea of Nelson and Guerra:

stochastic processes are related to the solutions of the Schrödinger equation. *It is the time development of the four-oscillator coherent states, which induces the related stochastic process,* and we shall discuss its role for the Kepler problem. A well known property of these states is that their time dependence $|\alpha, t\rangle = \exp(-i/\hbar\, H_0' t)$ can be transferred to the time dependence of the labels: $|\alpha, t\rangle = \eta(t)\, |a(t)\rangle$, where $\eta(t)$ is a phase factor ($\exp(-i\omega t/2)$ for a single harmonic oscillator), and $\alpha(t)$ is a solution of the classical equation of motion e.g.

$$\alpha(t) = \langle \alpha, t|\, a\, |\alpha, t\rangle = \langle \alpha|\, e^{(i/\hbar)H_0' t} a e^{-(i/\hbar)H_0' t}\, |\alpha\rangle \tag{4.1}$$

for the single harmonic oscillator variable.

Let us study the expression analogous to (4.1) in case of the Kepler problem

$$\alpha_j(t) := \langle \alpha|\, e^{(i/\hbar)Ht} a_j e^{-(i/\hbar)Ht}\, |\alpha\rangle = \langle \alpha|\, a_{H_j}(t)\, |\alpha\rangle . \tag{4.2}$$

Notice that $|\alpha, t\rangle = e^{-(i/\hbar)Ht}\, |\alpha\rangle$ is the Schrödinger type evolution of the coherent state. For the oscillator $\alpha_j(t)$ obeys the classical equation of motion, which is *not the case* for the Kepler problem, because of the tree approximation involved

$$\dot{\alpha}_j(t) = \frac{i}{\hbar} \langle \alpha|\, [H, a_{H_j}]\, |\alpha\rangle = \frac{i}{\hbar} \langle \alpha|\, :H, a_{H_j}]:\, |\alpha\rangle + \text{contractions}. \tag{4.3}$$

Let us notice that according to Sect. 3 solution of the equation

$$\dot{\alpha}_j^c(t) = \frac{i}{\hbar} \langle \alpha|\, :[H, a_{H_j}]:\, |\alpha\rangle \tag{4.4}$$

is the classical one. By virtue of (4.3) we find that $\alpha_j^c(t)$ is the tree approximation contribution to $\alpha_j(t)$

$$\alpha_j(t) = \alpha_j^c(t) + \cdots . \tag{4.5}$$

Now let us observe that if to formally replace the coherent state labels α_j in $|\alpha\rangle$ by $\alpha_j(t)$ where $\alpha_j(t)$ follows from (4.2) then we have

$$\alpha_j(t) = \langle \alpha(t)|\, a_j\, |\alpha(t)\rangle = \langle \alpha|\, e^{(i/\hbar)Ht} a_j e^{-(i/\hbar)Ht}\, |\alpha\rangle \tag{4.6}$$

which provides us with a quantally implemented time development of the state label $\alpha_i = \alpha_i(t)$, and thus makes it possible to take $|\alpha(t)\rangle$ as a starting point for the construction of the (related) stochastic process despite of the fact that $\langle x \mid \alpha(t)\rangle$ does not obey the Schrödinger equation for the original Kepler problem. According to (4.5) we have

$$\langle x \mid \alpha(t)\rangle = \langle x \mid \alpha^c(t)\rangle + \cdots \tag{4.7}$$

and $\langle x \mid \alpha^c(t)\rangle$ comes from the Schrödinger equation for the four-oscillator.

In below we shall confine our attention to the pure $\langle x \mid \alpha^c(t)\rangle$ contribution to $\langle x \mid \alpha(t)\rangle$, which should be viewed as the tree approximation only.

We know that the four-oscillator coherent state reads

$$|\alpha\rangle = \exp\left(-\frac{1}{2}\sum_{i=1}^{4} |\alpha_i|^2\right) \sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{(\alpha_1)^{n_1} (\alpha_2)^{n_2} (\alpha_3)^{n_3} (\alpha_4)^{n_4}}{(n_1!\, n_2!\, n_3!\, n_4!)^{1/2}} |n_1, n_2, n_3, n_4\rangle \tag{4.8}$$

and in the coordinate representation it equals to

$$\psi(x) := \langle x \mid \alpha \rangle$$
$$= \frac{m\omega}{\hbar\pi} \exp\left\{-i \operatorname{Re} \alpha \cdot \operatorname{Im} \alpha + i\sqrt{2}\, \frac{1}{b} \operatorname{Im} \alpha \cdot x - \frac{1}{2}\left(\frac{x}{b} - \sqrt{2} \operatorname{Re} \alpha\right)^2\right\},$$
$$b = \left(\frac{\hbar}{m\omega}\right)^{1/2} \tag{4.9}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, $x = (x_1, x_2, x_3, x_4)$. If to insert $\alpha_j^c(t)$ instead of the arbitrary parameters α_i we have (using the standard relations (2.17a))

$$\psi(x, t) = \langle x \mid \alpha^c(t) \rangle$$
$$= \frac{1}{(2\pi\sigma)^{1/4}} \exp\left\{-\frac{1}{4\sigma}\left(x - Q^c(t)\right)^2 + \frac{i}{\hbar}\, x \cdot P^c(t) - \frac{i}{\hbar}\, Q^c(t) \cdot P^c(t)\right\},$$
$$\sigma = \frac{\hbar}{2m\omega}. \tag{4.10}$$

Formally the state (4.10) is the same as the four-oscillator coherent state, but instead of the oscillator phase-space variables q^c, p^c the classical phase-space variables for the Kepler problem Q^c, P^c ($Q^c(t)$, $P^c(t)$ are the solutions of (3.7a)) are in use.

Following the oscillator pattern, we can in principle introduce four stochastic processes with the density-phase variables defined according to

$$\varrho_j(x_j, t) := \frac{1}{(2\pi\sigma)^{1/2}} \exp\left\{-\frac{1}{2\sigma}\left(x_j - Q_j^c(t)\right)^2\right\}$$
$$S_j(x_j, t) := x_j \cdot P_j^c(t) - \frac{1}{2} Q_j^c(t) \cdot P_j^c(t) - \frac{1}{2} \hbar\omega. \tag{4.11}$$

Because of constraints these variables are not independent. By virtue of (2.12) the Heisenberg operators obey

$$a_{H_1}^+(t)\, a_{H_1}(t) + a_{H_2}^+(t)\, a_{H_2}(t) - a_{H_3}^+(t)\, a_{H_3}(t) - a_{H_4}^+(t)\, a_{H_4} = 0 \tag{4.12}$$

while for the coherent state averages there holds

$$|\alpha_1(t)|^2 + |\alpha_2(t)|^2 - |\alpha_3(t)|^2 - |\alpha_4(t)|^2 = 0 \tag{4.13}$$

with $\alpha_j(t)$ given by (4.2). In the tree approximation we have the same restriction but for classical trajectories

$$|\alpha_1^c(t)|^2 + |\alpha_2^c(t)|^2 - |\alpha_3^c(t)|^2 - |\alpha_4^c(t)|^2 = 0. \tag{4.13}$$

Using the formulas

$$Q_j^c(t) = \int dx x \cdot \varrho_j(x, t) \tag{4.14a}$$

$$P_j^c(t) = \int dx \varrho_j(x, t)\, \nabla_x S_j(x, t) \tag{4.14b}$$

$$\alpha_3 = \left(\frac{P_3}{\hbar} - \frac{1}{2}\right)^{1/2} \exp(-iQ_3)$$
$$\alpha_4 = \left(\frac{P_1 - P_3}{\hbar} - \frac{1}{2}\right)^{1/2} \tag{4.21}$$

which determines the four-oscillator coherent state domain, displaying an explicit parametrization in terms of the phase space variables of the classical Kepler problem (apart from the presence of the Planck constant $\hbar$).

The above analysis at the same time implies the following transformation of the four-oscillator Hamiltonian

$$H_0'^c = \omega \sum_{i=1}^{4} J_i = 2\omega P_1. \tag{4.22}$$

By comparing H'^c and $H_0'^c$ in the $(Q_1, Q_2, Q_3, P_1, P_1, P_3)$ parametrization, we realize that the canonical transformation

$$P_1 \to \tilde{P}_1 = -\frac{2\pi^2 m(Ze^2)^2}{2\omega} \frac{1}{P_1^2}$$
$$Q_1 \to \tilde{Q}_1 = \frac{\omega}{2\pi^2 m(Ze^2)^2} Q_1 P_1^3$$
$$Q_i = \tilde{Q}_i \qquad i = 2, 3,$$
$$P_i = \tilde{P}_i \tag{4.23}$$

allows to identify the transformed four-oscillator Hamiltonian with the classical Kepler one

$$\tilde{H}_0'^c = 2\omega\tilde{P}_1 = H'^c \tag{4.24}$$

which establishes the link between the classical four-oscillator dynamics and the dynamics governed by the Kepler Hamiltonian: in the tilde parameterization the classical Kepler and (constrained) four-oscillator dynamics do coincide. It means that the classical dynamics of Kepler trajectories $Q_j^c(t)$, $P_j^c(t)$ can be either generated by the (classical) Kepler Hamiltonian, or equivalently by the (canonical transformed) constrained four-oscillator Hamiltonian.

It is the point at which we make transparent connection with the stochastic mechanics of Nelson and Guerra. Namely, since the coherent state expectation value of the four-oscillator Hamiltonian entirely fits to (1.1)—(1.13) the fact that the classical oscillator dynamics coincide (in another parametrization) with the Kepler dynamics allows to extend the stochastic mechanics methods, valid in the oscillator case, to the Kepler problem.

Let us denote ϱ, S of (4.11) as ϱ^K, S^K (K means Kepler) and for oscillator [3] ϱ^{osc}, S^{osc}, respectively. In both cases [$E > 0$ and $E < 0$) on the classical level we have a canonical transformation from the Kepler Hamiltonian to the four-oscillator one (repulsive and attractive, respectively). By virtue of (4.16)) (which is valid for continuous and bound systems) we find the canonical passage from (ϱ^K, S^K) to (ϱ^{osc}, S^{osc}). Let us emphasize that (ϱ^K, S^K) do not describe the stochastic process underlying the complete quantum Kepler problem. The equivalence with the four-oscillator arises as a result of the tree approximation procedure. One can state the problem of constructing the stochastic processes for the Kepler case itself. Our result can be verbalized as follows: in the tree approximation the stochastic mechanics of the Kepler problem is the same as this for the constrained four-oscillator.

Appendix A:

Poisson brackets (2.27): The Canonical Structure after the Kustaanheimo-Stiefel Transformation

For functions u, v on the classical four-oscillator phase-space we have

$$\{u, v\}_{Q,P} = \sum_{k=1}^{4} \left(\frac{\partial u}{\partial Q_k} \frac{\partial v}{\partial P_k} - \frac{\partial u}{\partial P_k} \frac{\partial v}{\partial Q_k} \right). \tag{A.1}$$

From (2.26) there follows

$$\begin{aligned} q_1 &= 2(Q_1Q_3 - Q_2Q_4) \\ q_2 &= 2(Q_1Q_2 + Q_3Q_4) \\ q_3 &= Q_2^2 + Q_3^2 - Q_1^2 - Q_4^2 \end{aligned} \tag{A.2a}$$

$$\begin{aligned} p_1 &= \frac{1}{2r} (Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4) \\ p_2 &= \frac{1}{2r} (Q_2P_1 + Q_1P_2 + Q_4P_3 + Q_3P_4) \\ p_3 &= \frac{1}{2r} (-Q_1P_1 + Q_2P_2 + Q_3P_3 - Q_4P_4). \end{aligned} \tag{A.2b}$$

The Poisson brackets $\{q_i, q_j\} = 0$ are trivial. To proceed with $\{p_i, p_j\}$ let us first notice that

$$\frac{\partial}{\partial Q_k} \frac{1}{2r} = -\frac{1}{r^2} Q_k \tag{A.3}$$

which implies

$$\begin{aligned} \{p_1, p_2\} = \sum_{k=1}^{4} \Big\{ &-\frac{1}{r^2} Q_k(Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4) \frac{\partial p_2}{\partial P_k} \\ &+ \frac{1}{r^2} \frac{\partial p_1}{\partial P_k} Q_k(Q_2P_1 + Q_1P_2 + Q_4P_3 + Q_3P_4) \\ &+ \frac{1}{2r} \frac{\partial}{\partial Q_k} (Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4) \frac{\partial p_2}{\partial P_k} \\ &- \frac{1}{2r} \frac{\partial p_1}{\partial P_k} \frac{\partial}{\partial Q_k} (Q_2P_1 + Q_1P_2 + Q_4P_3 + Q_3P_4) \Big\} \\ = &-\frac{1}{r^3} (Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4)(Q_2Q_1 + Q_3Q_4) \\ &+ \frac{1}{r^3} (Q_2P_1 + Q_1P_2 + Q_4P_3 + Q_3P_4)(Q_3Q_1 - Q_4Q_2) \\ &+ \frac{1}{2r^2} (Q_2P_3 - Q_3P_2 + Q_4P_1 - Q_1P_4) \end{aligned}$$

$$= -\frac{1}{r^3}(Q_3^2Q_4P_1 - Q_4^2Q_3P_2 + Q_1^2Q_2P_3 - Q_2^2Q_1P_4)$$
$$+ \frac{1}{r^3}(Q_1^2Q_3P_2 + Q_3^2Q_1P_4 - Q_2^2Q_4P_1 - Q_4^2Q_2P_3)$$
$$+ \frac{1}{2r^2}(Q_2P_3 - Q_3P_2 + Q_4P_1 - Q_1P_4)$$
$$= \frac{1}{r^3}[(Q_3^2 + Q_2^2)(Q_1P_4 - Q_4P_1) + (Q_1^2 + Q_4^2)(Q_3P_2 - Q_2P_3)]$$
$$+ \frac{1}{2r^2}(Q_2P_3 - Q_3P_2 + Q_4P_1 - Q_1P_4)$$
$$= \frac{1}{r^3}[(Q_3^2 + Q_2^2)(Q_1P_4 - Q_4P_1 + Q_2P_3 - Q_3P_2) + r(Q_3P_2 - Q_2P_3)]$$
$$+ \frac{1}{2r^2}(Q_2P_3 - Q_3P_2 + Q_4P_1 - Q_1P_4)$$
$$= \frac{1}{r^3}(Q_3^2 + Q_2^2)(Q_1P_4 - Q_4P_1 + Q_2P_3 - Q_3P_2)$$
$$- \frac{1}{2r^2}(Q_1P_4 - Q_4P_1 + Q_2P_3 - Q_3P_2) \approx 0, \tag{A.4a}$$

$$\{p_1, p_3\} = \sum_{k=1}^{4}\left\{-\frac{1}{r^2}Q_k(Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4)\frac{\partial p_3}{\partial P_k}\right.$$
$$+ \frac{1}{r^2}\frac{\partial p_1}{\partial P_k}Q_k(-Q_1P_1 + Q_2P_2 + Q_3P_3 - Q_4P_4)$$
$$+ \frac{1}{2r}\frac{\partial}{\partial Q_k}(Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4)\frac{\partial p_3}{\partial P_k}$$
$$\left.- \frac{1}{2r}\frac{\partial p_1}{\partial P_k}\frac{\partial}{\partial Q_k}(-Q_1P_1 + Q_2P_2 + Q_3P_3 - Q_4P_4)\right\}$$
$$= -\frac{1}{2r^3}(Q_3P_1 - Q_4P_2 + Q_1P_3 - Q_2P_4)(-Q_1^2 + Q_2^2 + Q_3^2 - Q_4^2)$$
$$+ \frac{1}{r^3}(Q_3Q_1 - Q_4Q_2)(-Q_1P_1 + Q_2P_2 + Q_3P_3 - Q_4P_4)$$
$$+ \frac{1}{2r^2}(Q_3P_1 + Q_4P_2 - Q_1P_3 - Q_2P_4)$$
$$= \frac{1}{2r^3}(-Q_1^2Q_3P_1 - Q_2^2Q_3P_1 - Q_3^2Q_3P_1 + Q_4^2Q_3P_1 - Q_1^2Q_4P_2$$
$$- Q_2^2Q_4P_2 + Q_3^2Q_4P_2 - Q_4^2Q_4P_2 + Q_1^2Q_1P_3 - Q_2^2Q_1P_3 + Q_3^2Q_1P_3$$
$$+ Q_4^2Q_1P_3 - Q_1^2Q_2P_4 + Q_2^2Q_2P_4 + Q_3^2Q_2P_4 + Q_4^2Q_2P_4 + 2Q_3Q_2Q_1P_2$$
$$- 2Q_4Q_3Q_1P_4 + 2Q_4Q_2Q_1P_1 - 2Q_4Q_3Q_2P_3)$$
$$+ \frac{1}{2r^3}(Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2)(Q_3P_1 + Q_4P_2 - Q_1P_3 - Q_2P_4)$$

$$= \frac{1}{r^3} (Q_4^2 Q_3 P_1 + Q_3^2 Q_4 P_2 - Q_2^2 Q_1 P_3 - Q_1^2 Q_2 P_4 + Q_3 Q_2 Q_1 P_2 - Q_3 Q_1 Q_4 P_4 + Q_4 Q_2 Q_1 P_1 - Q_4 Q_3 Q_2 P_3)$$

$$= \frac{1}{r^3} [(Q_3 Q_4 + Q_2 Q_1) Q_4 P_1 - (Q_3 Q_4 + Q_2 Q_1) Q_1 P_4 + (Q_3 Q_4 + Q_2 Q_1) Q_3 P_2 - (Q_3 Q_4 + Q_2 Q_1) Q_2 P_3]$$

$$= \frac{1}{r^3} (Q_3 Q_4 + Q_2 Q_1)(Q_4 P_1 - Q_1 P_4 + Q_3 P_2 - Q_2 P_3) \approx 0, \qquad \text{(A.4b)}$$

$$\{p_2, p_3\} = \sum_{k=1}^{4} \Big\{ -\frac{1}{r^2} Q_k (Q_2 P_1 + Q_1 P_2 + Q_4 P_3 + Q_3 P_4) \frac{\partial p_3}{\partial P_k} + \frac{1}{r^2} \frac{\partial p_2}{\partial P_k} Q_k (-Q_1 P_1 + Q_2 P_2 + Q_3 P_3 - Q_4 P_4) + \frac{1}{2r} \frac{\partial}{\partial Q_k} (Q_2 P_1 + Q_1 P_2 + Q_4 P_3 + Q_3 P_4) \frac{\partial p_3}{\partial P_k} - \frac{1}{2r} \frac{\partial p_2}{\partial P_k} \frac{\partial}{\partial Q_k} (-Q_1 P_1 + Q_2 P_2 + Q_3 P_3 - Q_4 P_4)$$

$$= -\frac{1}{2r^3} (Q_2 P_1 + Q_1 P_2 + Q_4 P_3 + Q_3 P_4)(-Q_1^2 + Q_2^2 + Q_3^2 - Q_4^2) + \frac{1}{r^3} (Q_2 Q_1 + Q_4 Q_3)(-Q_1 P_1 + Q_2 P_2 + Q_3 P_3 - Q_4 P_4) + \frac{1}{2r^2} (Q_2 P_1 - Q_1 P_2 + Q_3 P_4 - Q_4 P_3)$$

$$= \frac{1}{r^3} (-Q_3^2 Q_1 P_2 - Q_2^2 Q_4 P_3 + Q_1^2 Q_3 P_4 + Q_4^2 Q_2 P_1 - Q_1 Q_3 Q_4 P_1 + Q_2 Q_3 Q_4 P_2 + Q_1 Q_2 Q_3 P_3 - Q_1 Q_2 Q_4 P_4)$$

$$= \frac{1}{r^3} [(Q_2 Q_4 - Q_1 Q_3) Q_3 P_2 - (Q_4 Q_2 - Q_1 Q_3) Q_2 P_3 + (Q_4 Q_2 - Q_1 Q_3) Q_4 P_1 - (Q_4 Q_2 - Q_1 Q_3) Q_1 P_4]$$

$$= \frac{1}{r^3} (Q_2 Q_4 - Q_1 Q_3)(Q_4 P_1 - Q_1 P_4 + Q_3 P_2 - Q_2 P_3) \approx 0. \qquad \text{(A.4c)}$$

Everywhere $\approx$ means vanishing of the expression on the surface of constraint. The remaining brackets satysfy exact identities

$$\{q_1, p_1\} = \sum_{k=1}^{4} \frac{\partial q_1}{\partial Q_4} \frac{\partial p_1}{\partial P_k} = \frac{1}{r} (Q_3^2 + Q_4^2 + Q_1^2 + Q_2^2) = 1 \qquad \text{(A.5a)}$$

$$\{q_1, p_2\} = \sum_{k=1}^{4} \frac{\partial q_1}{\partial Q_k} \frac{\partial p_2}{\partial P_k} = \frac{1}{r} (Q_3 Q_2 - Q_4 Q_1 + Q_1 Q_4 - Q_2 Q_3) = 0 \qquad \text{(A.5b)}$$

$$\{q_1, p_3\} = \sum_{k=1}^{4} \frac{\partial q_1}{\partial Q_k} \frac{\partial p_3}{\partial P_k} = \frac{1}{r} (-Q_3 Q_1 - Q_4 Q_2 + Q_1 Q_3 + Q_2 Q_4) = 0 \qquad \text{(A.5c)}$$

$$\{q_2, p_1\} = \sum_{k=1}^{4} \frac{\partial q_2}{\partial Q_k} \frac{\partial p_1}{\partial P_k} = \frac{1}{r} (Q_2Q_3 - Q_1Q_4 + Q_4Q_1 - Q_3Q_2) = 0 \tag{A.5d}$$

$$\{q_2, p_2\} = \sum_{k=1}^{4} \frac{\partial q_2}{\partial Q_k} \frac{\partial p_2}{\partial P_k} = \frac{1}{r} (Q_2^2 + Q_1^2 + Q_4^2 + Q_3^2) = 1 \tag{A.5e}$$

$$\{q_2, p_3\} = \sum_{k=1}^{4} \frac{\partial q_2}{\partial Q_k} \frac{\partial p_3}{\partial P_k} = \frac{1}{r} (-Q_2Q_1 + Q_1Q_2 + Q_4Q_3 - Q_3Q_4) = 0 \tag{A.5f}$$

$$\{q_3, p_1\} = \sum_{k=1}^{4} \frac{\partial q_3}{\partial Q_k} \frac{\partial p_1}{\partial P_k} = \frac{1}{r} (-Q_1Q_3 - Q_2Q_4 + Q_3Q_1 + Q_4Q_2) = 0 \tag{A.5g}$$

$$\{q_3, p_2\} = \sum_{k=1}^{4} \frac{\partial q_3}{\partial Q_k} \frac{\partial p_2}{\partial P_k} = \frac{1}{r} (-Q_1Q_2 - Q_2Q_1 + Q_3Q_4 - Q_4Q_3) = 0 \tag{A.5h}$$

$$\{q_3, p_3\} = \sum_{k=1}^{4} \frac{\partial q_3}{\partial Q_k} \frac{\partial q_3}{\partial P_k} = \frac{1}{r} (Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) = 1. \tag{A.5i}$$

Appendix B:

Tree Approximation of the Heisenberg Equations

Let us begin from the derivation of (3.5) and (3.8). We have:

$$Y^2 H = \frac{2m(Ze^2)^2}{\hbar^2} \tag{B.1}$$

so that

$$[Y^2 H, a_k]_- = [Y^2 H, a_k^+]_- = 0, \qquad k = 1, 2, 3, 4. \tag{B.2}$$

Furthermore

$$Y^2[H, b]_- + [Y^2, b]_- H = 0, \qquad b = a_k, a_k^+ \tag{B.2a}$$

$$[H, a_k]_- = -\frac{1}{Y^2} [Y^2, a_k]_- H \qquad k = 1, 2, 3, 4 \tag{B.2b}$$

$$[H, a_k^+]_- = -\frac{1}{Y^2} [Y^2, a_k^+]_- H. \tag{B.2c}$$

Commutators $[Y^2, a_k]_-$, $[Y^2, a_k^+]_-$ are trivial

$$[Y^2, a_i]_- = -Y a_{5-i}^+ - a_{5-i}^+ Y \tag{B.3a}$$

$$[Y^2, a_i^+]_- = -Y a_{5-i} - a_{5-i} Y. \qquad i = 1, 2, 3, 4 \tag{B.3b}$$

Inserting (B.3) to (B.2) we obtain (3.5). Analogously for H', (3.8)

$$[H', a_k]_- = -\frac{1}{X} [X, a_k]_- H' \tag{B.4a}$$

$$[H', a_k^+]_- = -\frac{1}{X} [X, a_k^+]_- H' \qquad k = 1, 2, 3, 4 \tag{B.4b}$$

while

$$[X, a_i]_- = -(a_1^+a_1 + a_2^+a_2 + 1)\, a_i - a_i(a_1^+a_1 + a_2^+a_2 + 1), \qquad i = 1, 2 \tag{B.5a}$$

$$[X, a_i]_- = -(a_3^+a_3 + a_4^+a_4 + 1)\, a_i - a_i(a_3^+a_3 + a_4^+a_4 + 1), \qquad i = 3, 4 \tag{B.5b}$$

$$[X, a_i^+]_- = (a_1^+a_1 + a_2^+a_2 + 1)\, a_i^+ + a_i^+(a_1^+a_1 + a_2^+a_2 + 1), \qquad i = 1, 2 \tag{B.5c}$$

$$[X, a_i^+]_- = (a_3^+a_3 + a_4^+a_4 + 1)\, a_i^+ + a_i^+(a_3^+a_3 + a_4^+a_4 + 1) \qquad i = 3, 4. \tag{B.5d}$$

From (B.4) and (B.5) we get immediately (3.8).

We are now ready to pass to classical Hamiltonian equations. Let us begin from the continuum case

$$\begin{aligned}
\dot{Q}_1 &= \frac{dQ_1}{dt} = \langle\alpha|\, :\frac{d\hat{Q}_1}{dt}:\, |\alpha\rangle = \frac{i}{\hbar} \langle\alpha|\, :[H, \hat{Q}_1]_-:\, |\alpha\rangle \\
&= \frac{i}{\hbar}\,\frac{1}{2}\left(\frac{2\hbar}{m\omega}\right)^{1/2} \langle\alpha|\, :[H, a_1 + a_1^+]:\, |\alpha\rangle \\
&= \frac{i}{\hbar}\,\frac{1}{2}\left(\frac{2\hbar}{m\omega}\right)^{1/2} \langle\alpha|\, :\frac{1}{Y} a_4^+H + \frac{1}{Y^2} a_4^+YH - \frac{1}{Y} a_4H - \frac{1}{Y^2} a_4YH:\, |\alpha\rangle \\
&= -\frac{i}{\hbar}\left(\frac{2\hbar}{m\omega}\right)^{1/2} \frac{2m(Ze^2)^2}{\hbar^2} \langle\alpha|\, :(a_4 - a_4^+)\,\frac{1}{Y^3}:\, |\alpha\rangle \\
&= \frac{1}{2}\,\omega^2(Ze^2)^2 \frac{P_4}{\left[\frac{m\omega^2}{2}(Q_1Q_4 - Q_2Q_3) - \frac{1}{2m}(P_1P_4 - P_2P_3)\right]^3} = \frac{\partial H^c}{\partial P_1}
\end{aligned} \tag{B.6a}$$

$$\begin{aligned}
\dot{P}_1 &= \frac{dP_1}{dt} = \langle\alpha|\, :\frac{d\hat{P}_1}{dt}:\, |\alpha\rangle = \frac{i}{\hbar}\langle\alpha|\, :[H, \hat{P}_1]_-:\, |\alpha\rangle \\
&= \frac{1}{2\hbar}(2\hbar m\omega)^{1/2}\langle\alpha|\, :[H, a_1 - a_1^+]_-:\, |\alpha\rangle \\
&= \left(\frac{m\omega}{2\hbar}\right)^{1/2} \frac{4m(Ze^2)^2}{\hbar^2}\langle\alpha|\, :(a_4 + a_4^+)\,\frac{1}{Y^3}:\, |\alpha\rangle \\
&= \frac{1}{2}\,\omega^4m^2(Ze^2)^2 \frac{Q_4}{\left[\frac{m\omega^2}{2}(Q_1Q_4 - Q_2Q_3) - \frac{1}{2m}(P_1P_4 - P_2P_3)\right]^3} = -\frac{\partial H^c}{\partial Q_1}.
\end{aligned} \tag{B.6b}$$

The rest of equations (3.7) follows in precisely the same way. Let us recall that $:[H, \hat{Q}]_-:$ means first computing the commutator, and next the normal ordering of the result. For the discrete case ($E < 0$) we have (3.9) i.e.

$$\begin{aligned}
\dot{Q}_1 &= \frac{i}{\hbar}\,\frac{1}{2}\left(\frac{2\hbar}{m\omega}\right)^{1/2} \langle\alpha|\, :[H', a_1 + a_1^+]_-:\, |\alpha\rangle \\
&= \frac{i}{\hbar}\left(\frac{2\hbar}{m\omega}\right)^{1/2} \frac{m(Ze^2)^2}{\hbar^2}\langle\alpha|\, :(a_1 - a_1^+)\,(a_1^+a_1 + a_2^+a_2 + 1)\,\frac{1}{X^2}:\, |\alpha\rangle
\end{aligned}$$

$$= 2(Ze^2)^2\,\omega^2$$
$$\times \frac{P_1\left[\frac{m\omega^2}{2}(Q_1^2+Q_2^2)+\frac{1}{2m}(P_1^2+P_2^2)\right]}{\left\{\left[\frac{m\omega^2}{2}(Q_1^2+Q_2^2)+\frac{1}{2m}(P_1^2+P_2^2)\right]^2+\left[\frac{m\omega^2}{2}(Q_3^2+Q_4^2)+\frac{1}{2m}(P_3^2+P_4^2)\right]^2\right\}^2}$$
$$= \frac{\partial H'^c}{\partial P_1}, \tag{B.7a}$$

$$\dot{P}_1 = \frac{1}{2\hbar}(2\hbar m\omega)^{1/2}\langle\alpha|\,:[H', a_1 - a_1^+]_-:\,|\alpha\rangle$$
$$= -\frac{1}{\hbar}(2\hbar m\omega)^{1/2}\frac{m(Ze^2)^2}{\hbar^2}\langle\alpha|\,:(a_1+a_1^+)\,(a_1^+a_1+a_2^+a_2+1)\,\frac{1}{X^2}:\,|\alpha\rangle$$
$$= -2m^2\omega^4(Ze^2)^2$$
$$\times \frac{Q_1\left[\frac{m\omega^2}{2}(Q_1^2+Q_2^2)+\frac{1}{2m}(P_1^2+P_2^2)\right]}{\left\{\left[\frac{m\omega^2}{2}(Q_1^2+Q_2^2)+\frac{1}{2m}(P_1^2+P_2^2)\right]^2+\left[\frac{m\omega^2}{2}(Q_3^2+Q_4^2)+\frac{1}{2m}(P_3^2+P_4^2)\right]^2\right\}^2}$$
$$= -\frac{\partial H'^c}{\partial Q_1}. \tag{B.7b}$$

The same scheme works for the remaining equations (3.9).

Appendix C:

Classical-Quantum Relationship for the Repulsive Harmonic Oscillator

We have the following Hamiltonian for the repulsive harmonic four-oscillator

$$H_0 = \frac{1}{2m}\sum_{i=1}^{4}\hat{P}_i^2 - \frac{1}{2}m\omega^2\sum_{i=1}^{4}\hat{Q}_i^2. \tag{C.1}$$

Taking into account the boson realisation of operators $\hat{P}_i$, $\hat{Q}_i$ (2.14) we get

$$H_0 = -\frac{1}{2}\hbar\omega\sum_i(a_ia_i + a_i^+a_i^+). \tag{C.2}$$

We shall show that $\alpha_j(t)$ defined as in (4.2), with H_0 instead of H fulfil the classical equations of motion (such property is obviously shared by the attractive oscillator). We have:

$$\alpha_j(t) = \langle\alpha|\,e^{(i/\hbar)H_0t}a_je^{-(i/\hbar)H_0t}\,|\alpha\rangle \tag{C.3}$$

and

$$[H_0, a_j]_- = \hbar\omega a_j^+ \tag{C.4a}$$

$$[H_0, a_j^+]_- = -\hbar\omega a_j, \qquad j = 1, 2, 3, 4. \tag{C.4b}$$

Using the formula

$$e^XYe^{-X} = X + [Y, X]_- + \frac{1}{2!}[Y, [Y, X]_-]_- + \cdots \tag{C.5}$$

we arrive at

$$e^{(i/\hbar)H_0t}a_je^{-(i/\hbar)H_0t} = a_j + \frac{i}{\hbar}t[H_0, a_j]_- + \left(\frac{i}{\hbar}t\right)^2\frac{1}{2!}[H_0, [H_0, a_j]_-]_- + \cdots$$
$$= a_j + i\omega + a_j^+ - (i\omega t)^2\frac{1}{2!}a_j - (i\omega t)^3\frac{1}{3!}a_j^+ + (i\omega t)^4\frac{1}{4!}a_j + (i\omega t)^5\frac{1}{5!}a_j^+ + \cdots$$
$$= \left(1 + (\omega t)^2\frac{1}{2!} + (\omega t)^4\frac{1}{4!} + (\omega t)^6\frac{1}{6!} + \cdots\right)a_j + i\left(\omega t + (\omega t)^3\frac{1}{3!} + (\omega t)^5\frac{1}{5!} + \cdots\right)a_j^+$$
$$= a_j\cosh\omega t + ia_j^+\sinh\omega t, \tag{C.6a}$$

$$e^{(i/\hbar)H_0t}a_j^+e^{-(i/\hbar)H_0t} = a_j^+ + \frac{i}{\hbar}t[H_0, a_j^+]_- + \left(\frac{i}{\hbar}t\right)^2\frac{1}{2!}[H_0, [H_0, a_j^+]_-]_- + \cdots$$
$$= a_j^+ - i\omega ta_j - (i\omega t)^2\frac{1}{2!}a_j^+ + (i\omega t)^3\frac{1}{3!}a_j + (i\omega t)^4\frac{1}{4!}a_j^+ - (i\omega t)^5\frac{1}{5!}a_j - (i\omega t)^6\frac{1}{6!}a_j^+ + \cdots$$
$$= \left(1 + (\omega t)^2\frac{1}{2!} + (\omega t)^4\frac{1}{4!} + (\omega t)^6\frac{1}{6!} + \cdots\right)a_j^+ - i\left(\omega t + (\omega t)^3\frac{1}{3!} + (\omega t)^5\frac{1}{5!} + \cdots\right)a_j$$
$$= a_j^+\cosh\omega t - ia_j\sinh\omega t \tag{C.6b}$$

so that

$$\alpha_j(t) = \langle\alpha, t|\, a_j\, |\alpha, t\rangle = \alpha_j\cosh\omega t + i\bar{\alpha}_j\sinh\omega t \tag{C.7a}$$

$$\bar{\alpha}_j(t) = \langle\alpha, t|\, a_j^+\, |\alpha, t\rangle = \bar{\alpha}_j\cosh\omega t - i\alpha_j\sinh\omega t, \qquad j = 1, 2, 3, 4. \tag{C.7b}$$

After accounting for (2.14) and (2.17a) we find that:

$$P_k(t) = \langle\alpha, t|\, \hat{P}_k\, |\alpha, t\rangle = P_k^0\cosh\omega t + m\omega Q_k^0\sinh\omega t \tag{C.8a}$$

$$Q_k(t) = \langle\alpha, t|\, \hat{Q}_k\, |\alpha, t\rangle = Q_k^0\cosh\omega t + \frac{1}{m\omega}P_k^0\sinh\omega t, \qquad k = 1, 2, 3, 4. \tag{C.8b}$$

which on the other side follows from the Hamilton equations for the repulsive oscillator

$$\frac{dQ_k(t)}{dt} = \frac{1}{m}P_k(t) \tag{C.9a}$$

$$\frac{dP_k(t)}{dt} = m\omega^2Q_k(t), \qquad k = 1, 2, 3, 4. \tag{C.9b}$$

Accordingly (compare e.g. also the case of the attractive oscillator) we can construct the oscillator stochastic process directly (without use of any approximation) for the repulsive case. We have the following density-phase variables

$$\varrho_k(X_{k_j}t) := \frac{1}{(2\pi\sigma)^{1/2}} \exp\left\{-\frac{1}{2\sigma}\left(X_k - Q_{k_i}(t)\right)^2\right\} \tag{C.10a}$$

$$S_k(X_k, t) := x_k \cdot P_k(t) - \frac{1}{2} Q_k(t) \cdot P_k(t) - \frac{1}{2}\hbar\omega t, \qquad k = 1, 2, 3, 4, \tag{C.10b}$$

where Q_k, P_k are given by (C.8). The corresponding stochastic equations for the repulsive oscillator do not differ in form from this for the attractive one [3].

Appendix D:

Systematics of the Poisson Brackets

We have the following relationship between variables Q, P and α, $\bar{\alpha}$

$$\alpha_j = \left(\frac{m\omega}{2\hbar}\right)^{1/2} Q_j + \frac{i}{(2\hbar m\omega)^{1/2}} P_j \tag{D.1a}$$

$$\bar{\alpha}_j = \left(\frac{m\omega}{2\hbar}\right)^{1/2} Q_j - \frac{i}{(2\hbar m\omega)^{1/2}} P_j, \qquad j = 1, 2, 3, 4, \tag{D.1b}$$

so that

$$\{\alpha_k, \alpha_j\}_{Q,P} = \{\bar{\alpha}_k, \bar{\alpha}_j\}_{Q,P} = 0 \tag{D.2a}$$

$$\{\alpha_k, \bar{\alpha}_j\}_{Q,P} = \frac{1}{i\hbar} \delta_{kj} \tag{D.2b}$$

where

$$\{\cdot, \cdot\}_{Q,P} = \sum_{k=1}^{4} \left(\frac{\partial}{\partial Q_k}\frac{\partial}{\partial P_k} - \frac{\partial}{\partial P_k}\frac{\partial}{\partial Q_k}\right), \qquad k, j = 1, 2, 3, 4. \tag{D.3}$$

If we take

$$\{\cdot, \cdot\}_{\alpha,\bar{\alpha}} := \sum_{k=1}^{4} \left(\frac{\partial}{\partial \alpha_k}\frac{\partial}{\partial \bar{\alpha}_k} - \frac{\partial}{\partial \bar{\alpha}_k}\frac{\partial}{\partial \alpha_k}\right) \tag{D.4}$$

as the Poisson bracket definition, then

$$\{Q_k, Q_j\}_{\alpha,\bar{\alpha}} = \{P_k, P_j\}_{\alpha,\bar{\alpha}} = 0 \tag{D.5a}$$

$$\{Q_k, P_j\}_{\alpha,\bar{\alpha}} = i\hbar\delta_{kj}, \qquad k, j = 1, 2, 3, 4, \tag{D.5b}$$

and

$$\{\mathcal{A}, \mathcal{B}\}_{\alpha,\bar{\alpha}} = i\hbar\{\mathcal{A}, \mathcal{B}\}_{Q,P} \tag{D.6}$$

for two arbitrary functions of variables α, $\bar{\alpha}$ (respectively, by using (D.1) of variables Q, P).

Now let us pass to new canonical variables Q', P'. Because of

$$\{\mathcal{A}, \mathcal{B}\}_{Q',P'} = \sum_{s,l=1}^{4} \frac{\partial\mathcal{A}}{\partial\alpha_s}\frac{\partial\mathcal{B}}{\partial\alpha_l}\{\alpha_s, \alpha_l\}_{Q',P'} + \sum_{s,l=1}^{4} \frac{\partial\mathcal{A}}{\partial\bar{\alpha}_s}\frac{\partial\mathcal{B}}{\partial\bar{\alpha}_l}\{\bar{\alpha}_s, \bar{\alpha}_l\}_{Q',P'} + \sum_{s,l=1}^{4}\left(\frac{\partial\mathcal{A}}{\partial\alpha_s}\frac{\partial\mathcal{B}}{\partial\bar{\alpha}_l} - \frac{\partial\mathcal{A}}{\partial\bar{\alpha}_l}\frac{\partial\mathcal{B}}{\partial\alpha_s}\right)\{\alpha_s, \bar{\alpha}_l\}_{Q',P'} \tag{D.7}$$

and (D.6), since $(Q, P) \leftrightarrow (Q', P')$ is to be canonical, the following relations must hold true

$$\{\alpha_k, \alpha_j\}_{Q',P'} = \{\bar{\alpha}_k, \bar{\alpha}_j\}_{Q',P'} = 0 \tag{D.8}$$

$$\{\alpha_k, \bar{\alpha}_j\}_{Q',P'} = \frac{1}{i\hbar}\,\delta_{kj}\,, \qquad k, j = 1, 2, 3, 4.$$

Coming back to (4.16) we realize that

$$\{\alpha_k, \alpha_j\}_{\theta,J} = \{\bar{\alpha}_k, \bar{\alpha}_j\}_{\theta,J} = 0 \tag{D.9}$$

$$\{\alpha_k, \bar{\alpha}_j\}_{\theta,J} = \frac{1}{i\hbar}\,\delta_{kj}\,, \qquad k, j = 1, 2, 3, 4.$$

and comparing it with (D.8) we get

$$\{\mathcal{A}, \mathcal{B}\}_{\theta,J} = \{\mathcal{A}, \mathcal{B}\}_{Q,P}. \tag{D.10}$$

Now using formulas (D.1) and (4.14) we can express α, $\bar{\alpha}$ as a functions of ϱ and S, and find Poisson brackets (4.17) for the former in terms of the latter.

We have

$$\frac{\delta\alpha_j}{\delta\varrho_k(x)} = \delta_{jk}\left[\left(\frac{m\omega}{2\hbar}\right)^{1/2} x + \frac{i}{(2\hbar m\omega)^{1/2}}\,P_k\right] \tag{D.11a}$$

$$\frac{\delta\alpha_j}{\delta S_k(x)} = \delta_{jk}\,\frac{1}{\sigma}\,\frac{i}{(2\hbar m\omega)^{1/2}}\,\varrho_k(x)\,(x - Q_k) \tag{D.11b}$$

so that

$$\{\alpha_i, \alpha_j\}_{\varrho,S} = \{\bar{\alpha}_i, \bar{\alpha}_j\}_{\varrho,S} = 0, \qquad i, j = 1, 2, 3, 4. \tag{D.12}$$

For $\{\alpha_k, \bar{\alpha}_j\}_{\varrho,S}$ we find

$$\{\alpha_k, \bar{\alpha}_j\}_{\varrho,S} = \sum_{l=1}^{4} \int dx \left(\frac{\delta\alpha_k}{\delta\varrho_l(x)}\,\frac{\delta\bar{\alpha}_j}{\delta S_l(x)} - \frac{\delta\alpha_k}{\delta S_l(x)}\,\frac{\delta\bar{\alpha}_j}{\delta\varrho_l(x)}\right)$$
$$= \frac{1}{i\hbar}\,\frac{1}{\sigma}\,\delta_{kj}\int dx\varrho_k(x)\,(x^2 - Q_k\cdot x) = \frac{1}{i\hbar}\,\delta_{kj}\,, \qquad k, j = 1, 2, 3, 4, \tag{D.13}$$

and finally

$$\{\mathcal{A}, \mathcal{B}\}_{Q,P} = \{\mathcal{A}, \mathcal{B}\}_{\varrho,S}. \tag{D.14}$$

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