Heavy-tailed targets and (ab)normal asymptotics in diffusive motion

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Heavy-tailed asymptotics of pdfs induced by:

- Langevin equation with additive Lévy noise
- Lévy-Schrödinger semigroups (symmetric stable driver)
- diffusion-type processes (Wiener noise response to specific logarithmic potentials)

Issues addressed:

- differences/affinites in dynamical behavior
- common asymptotic stationary probability densities
- confinement (pdf has a finite number of moments)
- hyper-confinement (all moments in existence)
- (ab)normal (heavy-tailed) **thermalization** in Brownian motion
- transient diffusion: gaussian into heavy-tailed pdf

Contexts:

- **Mathematics and mathematical physics**: hypercontractive, intrinsically ultracontractive etc. semigroups, spectral properties of generators (and generalized Hamiltonians), various inequalities and eigenvalue plus eigenfunction estimates: lowest eigenvalue and the ground state

- **polymer physics:** topologically-induced „superdiffusions" and the likes

- **random search problem** (like e.g. animal foraging), Lévy flights in inhomogeneous media; incompete knowledge of search targets

- **computer-assisted issues:** various versions of truncated Levy flights, cut-offs removal, convergence in law

- **optical lattices:** transient diffusive dynamics (heavy- tailed asymptotics in Brownian motion), logarithmic potentials and „cooling forces"

„Rough” guide I: fractional semigroups

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Cauchy noise and affiliated stochastic processes

(P.G. + R.O.)

Note !

$$\hat{H} = -D\Delta + \mathcal{V} \longrightarrow \hat{H}_\mu \doteq \lambda |\Delta|^{\mu/2} + \mathcal{V}$$

$\exp(-t\hat{H})$ $\mu = 1$

t Є [0,T]

(cf. „ill - posed problems”)

$$\partial_t \theta_* = -|\nabla|\theta_* - V\theta_*, \quad \partial_t \theta = |\nabla|\theta + V\theta, \tag{21}$$

where V is a measurable function such that:

(a) for all $x \in R$, $V(x) \geqslant 0$,

(b) for each compact set $K \subset R$ there exists C_K such that for all $x \in K$, V is locally bounded $V(x) \leqslant C_K$.

Lemma 5: If $1 \leqslant r \leqslant p \leqslant \infty$ and $t > 0$, then the operators T_t^V defined by

$$(T_t^V f)(x) = E_x^C \left\{ f(X_t^C) \exp\left[-\int_0^t V(X_s^C) ds \right] \right\}$$

are bounded from $L^r(R)$ into $L^p(R)$. Moreover, for each $r \in [1,\infty]$ and $f \in L^r(R)$, $T_t^V f$ is a bounded and continuous function.

Lemma 7: For any $p \in [1,\infty]$ and $f \in L^p(R)$ there holds

$$(T_t^V f)(x) = \int_R k_t^V(x,y) f(y) dy,$$

where $k_t^V(x,y) \geqslant 0$ almost everywhere

Lemma 8: $k_t^V(x,y)$ is jointly continuous in (x,y).

Lemma 9: $k_t^V(x,y)$ is strictly positive.

$$\partial_t \theta_* = -|\nabla| \theta_* - V\theta_* , \qquad \partial_t \theta = |\nabla| \theta + V\theta$$

let $\rho_0(x)$ and $\rho_T(x)$ be strictly

t Є [0,T]

positive densities. Then, the Markov process X_t^V characterized by the transition probability density:

$$p^V(y,s,x,t) = k_{t-s}^V(x,y) \frac{\theta(x,t)}{\theta(y,s)} \tag{23}$$

and the density of distributions

$$\rho(x,t) = \theta_*(x,t)\theta(x,t),$$

where

$$\theta_*(x,t) = \int_R k_t^V(x,y) f(y) dy, \qquad \theta_*(y,t) = \int_R k_{T-t}^V(x,y) g(x) dx$$

is precisely that interpolating Markov process to which Theorem 1 extends its validity, when the perturbed semigroup kernel replaces the Cauchy kernel.

Clearly, for all $0 \leqslant s \leqslant t \leqslant T$ we have

$$\theta_*(x,t) = \int_R k_{t-s}^V(x,y)\theta_*(y,s) dy, \qquad \theta(y,s) = \int_R k_{t-s}^V(x,y)\theta(x,t) dx \tag{24}$$

ANALYTIC PROPERTIES OF FRACTIONAL SCHRÖDINGER SEMIGROUPS AND GIBBS MEASURES FOR SYMMETRIC STABLE PROCESSES

KAMIL KALETA AND JÓZSEF LŐRINCZI

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Definition 3.2 (Fractional Schrödinger operator for bounded potential). If $V \in L^\infty(\mathbf{R}^d)$ we call

$$H_\alpha := (-\Delta)^{\alpha/2} + V, \quad 0 < \alpha < 2 \tag{3.1}$$

fractional Schrödinger operator with potential V. We call the one-parameter operator semigroup $\{e^{-tH_\alpha} : t \geq 0\}$ *fractional Schrödinger semigroup.*

Theorem 3.1. (Functional integral representation) *Let* $V \in L^\infty(\mathbf{R}^d)$, *and* $f, g \in L^2(\mathbf{R}^d)$. *We have*

$$(f, e^{-t((-\Delta)^{\alpha/2}+V)}g) = \int_{\mathbf{R}^d} dx \mathbf{E}^x \left[\overline{f(X_0)} g(X_t) e^{-\int_0^t V(X_s)ds} \right]. \tag{3.2}$$

Note: fractional Kato class

$\alpha < d$ $\qquad \Pi_\alpha(y-x) = \int_0^\infty p(t, y-x)dt = \mathcal{A}_{d,\alpha}|y-x|^{\alpha-d}, \quad x, y \in \mathbf{R}^d.$ $\qquad \mathcal{A}_{d,\gamma} = 2^{-\gamma}\pi^{-d/2}\Gamma((d-\gamma)/2)|\Gamma(\gamma/2)|^{-1}$

$\alpha \geq d$ $\qquad \alpha = d = 1$ $\qquad \Pi_\alpha(x) = \frac{1}{\pi}\log\frac{1}{|x|}$

$\alpha > d = 1.$ $\qquad \Pi_\alpha(x) = (2\Gamma(\alpha)\cos(\pi\alpha/2))^{-1}|x|^{\alpha-1}, \qquad x \in \mathbf{R}^d$

Definition 3.1. (Fractional Kato-class) We say that the Borel function $V : \mathbf{R}^d \to \mathbf{R}$ belongs to the *fractional Kato-class* $\mathcal{K}^\alpha$ if V satisfies either of the two equivalent conditions

$$\lim_{\epsilon \to 0} \sup_{x \in \mathbf{R}^d} \int_{|y-x|<\epsilon} |V(y)\Pi_\alpha(y-x)|dy = 0,$$

$$\lim_{t \to 0} \sup_{x \in \mathbf{R}^d} \int_0^t (P_s|V|)(x)ds = 0.$$

We write $V \in \mathcal{K}^\alpha_{\mathrm{loc}}$ if $V\mathbf{1}_B \in \mathcal{K}^\alpha$ for every ball $B \subset \mathbf{R}^d$. Moreover, we say that V is a *fractional Kato-decomposable potential* whenever

$$V = V_+ - V_- \quad \text{with} \quad V_- \in \mathcal{K}^\alpha, \;\; V_+ \in \mathcal{K}^\alpha_{\mathrm{loc}},$$

where V_+ and V_- denote the positive and negative parts of V, respectively.

Example 3.1. Some examples and counterexamples of Kato-potentials are as follows.

(1) *Locally bounded potentials:* Let $V \in L^\infty_{\text{loc}}(\mathbf{R}^d)$. Then for all $\alpha \in (0,2)$ we have $V \in \mathcal{K}^\alpha_{\text{loc}}$ and V is Kato-decomposable.

(2) *Locally integrable potentials:* Let $\alpha \in (0,2)$. Then $\mathcal{K}^\alpha_{\text{loc}} \subset L^1_{\text{loc}}(\mathbf{R}^d)$.

Next we state and prove the existence and basic properties of the kernel for the semigroup $\{T_t : t \geq 0\}$.

Theorem 3.3. *Let V be a Kato-decomposable potential. The following properties hold:*

(1) *for every fixed $t > 0$ the operator T_t has a bounded integral kernel $u(t,x,y)$, i.e. $T_t f(x) = \int_{\mathbf{R}^d} u(t,x,y) f(y) dy$, $t > 0$, $x \in \mathbf{R}^d$, $f \in L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$;*

(2) *$u(t,x,y) = u(t,y,x)$, for every $t > 0$, $x, y \in \mathbf{R}^d$;*

(3) *for every $t > 0$, $u(t,x,y)$ is continuous on $\mathbf{R}^d \times \mathbf{R}^d$;*

(4) *$u(t,x,y)$ is strictly positive on $(0,\infty) \times \mathbf{R}^d \times \mathbf{R}^d$;*

(5) *for all $x, y \in \mathbf{R}^d$ and $s, t \in \mathbf{R}$, $s < t$, the functional representation*

$$u(t-s,x,y) = \int e^{-\int_s^t V(X_r(\omega))dr} d\nu^{x,y}_{[s,t)}(\omega), \tag{3.7}$$

holds, where the α-stable bridge measure $\nu^{x,y}_{[s,t)}$ is given by (2.7).

Assumption 4.1. Let $\lambda_0 := \inf \operatorname{Spec} H_\alpha$ be an isolated eigenvalue. We assume that the corresponding eigenfunction φ_0 such that $\|\varphi_0\|_2 = 1$, called *ground state*, exists.

Definition 6.1 (Fractional $P(\phi)_1$-process). We call the process $(\widetilde{X}_t, \mu^x)_{t\in\mathbf{R}}$ obtained in Theorem 6.1 the *fractional $P(\phi)_1$-process* related to the Kato-decomposable potential V. We will also refer to the measure μ on $(\Omega, \mathcal{F})$ with

$$\mu(A) = \int_{\mathbf{R}^d} \mathbf{E}_{\mu^x}[\mathbf{1}_A]\, \varphi_0^2(x)dx$$

as the *fractional $P(\phi)_1$-measure* corresponding to the Kato decomposable potential V.

For simplicity, we drop "fractional" in the use of the above terminology. For our purposes below it will be useful to see the measure μ as the measure with respect to the stable bridge.

Lemma 6.4. *We have for $A \in \mathcal{F}_{[s,t)}$, $s,t \in \mathbf{R}$,*

$$\mu(A) = \int_{\mathbf{R}^d} dx\varphi_0(x) \int_{\mathbf{R}^d} dy\varphi_0(y) \int_\Omega e^{-\int_s^t (V(X_r(\omega))-\lambda_0)dr} \mathbf{1}_A d\nu_{[s,t)}^{x,y}(\omega). \tag{6.11}$$

From Physica **A 389**, (2010), 4419, P. G. + V. S.:

The fractional analog of the generalized diffusion equation (2) reads: $\partial_t \Psi = -\hat{H}_\mu \Psi = -\lambda|\Delta|^{\mu/2}\Psi - \mathcal{V}(x)\Psi$. Looking for its stationary solutions, we realize that if a square root of a positive invariant pdf $\Psi \sim \rho_*^{1/2}$ is asymptotically to come out, then the fractional Sturm–Liouville operator should be used to derive an explicit form of $\rho_*^{1/2}$ for a given $\mathcal{V}$.

In the opposite situation, when $\rho_*(x)$ is a priori prescribed, we can determine $\mathcal{V}$ through a compatibility condition:

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2}\rho_*^{1/2}}{\rho_*^{1/2}}. \tag{7}$$

More math lore: (Kaleta, Kulczycki, Potential Analysis, (2010))

$$-(-\Delta)^{\alpha/2} - q \text{ in } \mathbf{R}^d, \text{ for } q \geq 0, \alpha \in (0,2)$$

Lemma 1. *Let $q \in L^\infty_{\mathrm{loc}}$, $q \geq 0$. If $q(x) \to \infty$ as $|x| \to \infty$ then for all $t > 0$ operators T_t are compact.*

Let us assume that for all $t > 0$ operators T_t are compact. The semigroup (T_t) is called *intrinsically ultracontractive* (abbreviated as IU) if for each $t > 0$ there is a constant $C_{q,t}$ such that

$$u(t,x,y) \leq C_{q,t}\, \varphi_1(x)\varphi_1(y), \quad x, y \in \mathbf{R}^d.$$

Theorem 3. *Let $q \in L^\infty_{\mathrm{loc}}$, $q \geq 0$. If $\lim_{|x|\to\infty} \frac{q(x)}{\log|x|} = \infty$, then the operators T_t are compact and the semigroup (T_t) is intrinsically ultracontractive.*

Theorem 4. *Let $q \in L^\infty_{\mathrm{loc}}$, $q \geq 0$ and $q(x) \to \infty$ as $|x| \to \infty$. If the semigroup (T_t) is intrinsically ultracontractive, then for any $\epsilon \in (0,1]$ we have $\lim_{|x|\to\infty} \frac{\sup_{y\in B(x,\epsilon)} q(y)}{\log|x|} = \infty$.*

„Rough” guide II: gradient perturbations

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Ornstein–Uhlenbeck–Cauchy process

(P.G. + R.O.)

It is worth noting that when the transition function is stochastically continuous (see Sec. IV B), then the corresponding semigroup T_t in $C_0(\mathbf{R})$ defined by

$$(T_t f)(x) = \int_{-\infty}^{\infty} p_t(y|x) f(y) dy \tag{21}$$

is strongly continuous, and so its generator L is densely defined.

In such a case we can also define an adjoint semigroup T_t^* acting on the space of (probability) densities $L^1(\mathbf{R}, dx)$,

$$(T_t^* \rho)(u) = \int_{-\infty}^{\infty} p_t(u|v) \rho(v) dv. \tag{22}$$

Its generator we denote by L^*.

$L^* = L_0 - \nabla(b\ \cdot)$ $L = L_0 + b\nabla$ $L_0 = |\nabla|$

$L = L_0 + b\nabla$ $L^* = L_0 - \nabla(b\ \cdot)$ $L_0 = |\nabla|$ $b(v) = -\lambda v$

transition probability function of the process $\mathbf{u}(t)$ satisfies the backward equation

$$\frac{\partial p_t(u|v)}{\partial t} = L_0 p_t(u|\cdot)(v) + b(v)\nabla_v p_t(u|v)$$

and the forward equation (the Fokker–Planck equation analog)

$$\frac{\partial p_t(u|v)}{\partial t} = L_0 p_t(\cdot|v)(u) - \nabla_u[b(u)p_t(u|v)].$$

Estimates of the Green function for the fractional Laplacian perturbed by gradient

Krzysztof Bogdan, Tomasz Jakubowski *†

September 14, 2010

Following [12] we let $\alpha \in (1,2)$. We will consider dimension $d \in \{2,3,\dots\}$, a nonempty bounded open $C^{1,1}$ set $D \subset \mathbb{R}^d$, its Green function G_D for $\Delta^{\alpha/2}$, and the Green function $\tilde{G}_D$ of the operator

$$L = \Delta^{\alpha/2} + b(x)\cdot\nabla ,$$

where b is a function in Kato class $\mathcal{K}_d^{\alpha-1}$ (for details see Section 2). Our interest in L is motivated by the development of the classical theory of the Laplacian, non-symmetry of L (we have $L^* = \Delta^{\alpha/2} - b(x)\cdot\nabla - \operatorname{div} b$), the fact that drift is quite a problematic addition to a jump type process, and by a handful of techniques which already exist for $\Delta^{\alpha/2}$

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Estimates of Heat Kernel of Fractional Laplacian Perturbed by Gradient Operators

Krzysztof Bogdan*, Tomasz Jakubowski*

Let d be a natural number, $\alpha \in (1, 2)$, and let $b = (b_j)_{j=1}^d : \mathbb{R}^d \to \mathbb{R}^d$ be a function in a Kato class $\mathcal{K}_d^{\alpha-1}$ defined below. Our aim is to construct and estimate the semigroup with (weak) generator $\Delta^{\alpha/2} f(x) + \sum_{j=1}^d b_j(x)\partial_j f(x)$.

Theorem 1. *There is a continuous transition density $p'(t, x, y)$ such that*

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} \frac{P'_t f(x) - f(x)}{t} g(x)\, dx = \int_{\mathbb{R}^d} \Big(\Delta^{\alpha/2} f(x) + b(x) \cdot \nabla f(x)\Big) g(x)\, dx, \quad (1)$$

where $f, g \in C_c^\infty(\mathbb{R}^d)$, and $P'_t f(x) = \int_{\mathbb{R}^d} p'(t, x, y) f(y)\, dy$.

Point od departure: standard Brownian motion

$$\dot{x} = b(x,t) + A(t)$$

$$\langle A(s) \rangle = 0 \qquad \langle A(s)A(s') \rangle = 2D\,\delta(s-s')$$

$D \doteq k_B T / m\beta$

$$\partial_t \rho = D\triangle\rho - \nabla\,(b \cdot \rho)$$ Fokker-Planck eq.

Smoluchowski diffusion processes

stationary asymptotic regime

$$b = \frac{f}{m\beta} = -\frac{1}{m\beta}\nabla V$$

$$b = \quad b_* = u_* = D\nabla \ln \rho_* \,.$$

Stationary pdf (Gibbs-Boltzmann form)

$$\rho_*(x) = \exp\left([F_* - V(x)]/k_B T\right) \doteq \exp[2\Phi(x)]$$

$$\rho_*^{1/2} = \exp\Phi \text{ and } b = 2D\nabla\Phi$$

Becoming parabolic - no difference in the ultimate dynamics and asymptotics of **the inferred pdf** !

$$\rho(x,t) \doteq \theta_*(x,t)\exp[\Phi(x)]\,.$$

Semigroup dynamics

$$\partial_t \theta_* = D\Delta\theta_* - \mathcal{V}\theta_*$$

$D = k_B T / m\beta$

$$\partial_t \theta = -D\Delta\theta + \mathcal{V}\theta$$ = 0 !

$$\mathcal{V}(x) = \frac{1}{2}\left(\frac{b^2}{2D} + \nabla b\right)$$

$$= D\,\frac{\Delta\rho_*^{1/2}}{\rho_*^{1/2}}\,.$$

$$\theta = \theta(x) = \exp\Phi(x)$$

Semigroup potential

$$\theta \sim \rho_*^{1/2}$$

pdf dynamics

$$\rho(x,t) \doteq \theta(x,t)\theta_*(x,t) = \int p(y,s,x,t)\rho(y,s)dy$$

F-P equation

$$\partial_t \rho = D\triangle\rho - \nabla\,(b\cdot\rho)\,.$$

Schrödinger semigroups

$$\theta_*(t) = [\exp(-t\hat{H})\theta_*](0)$$

$$\hat{H} = -D\Delta + \mathcal{V}$$

Note: suitable restrictons upon the semigroup potential need to be respected, to have a positive and continuous semigroup kernel function

$$k(y,s,x,t) = \left(\exp[-(t-s)\hat{H}]\right)(y,x) \quad = \int exp[-\int_s^t \mathcal{V}(X(u),u)du]\, d\mu[s,y \mid t,x]$$

$$\rho(x,t) \doteq \int p(y,s,x,t)\rho(y,s)dy$$

$$\rho(x,t) \doteq \theta_*(x,t)\exp[\Phi(x)]\,.$$

$$k(y,s,x,t) = p(y,s,x,t)\frac{\rho_*^{1/2}(y)}{\rho_*^{1/2}(x)} = p(y,s,x,t)\ \exp[\Phi(y) - \Phi(x)]$$

If $\rho_*(x)$ has the Gibbs form (actually, **Gibbs-Boltzmann**)

then $\Phi(y) - \Phi(x) = (1/2k_BT)[V(x) - V(y)]$

$$b(x) = -\nabla V(x)/(m\gamma)$$

V. Betz, J. Lörinczi, (2003**); ground state processes**, „relative to Brownian motion"

Given a Schrödinger operator with Kato decomposable potential V and ground state ψ_0, we define a probability measure μ on $(\Omega, \mathcal{F})$ (i.e., a stochastic process) by putting

$$\mu(A) = \int dx\, \psi_0(x) \int dy\, \psi_0(y) \int 1_A(\omega) e^{-\int_{-T}^{T} V(\omega_s)\,ds}\, d\mathcal{W}_T^{x,y}(\omega) \qquad (2.7)$$

$$e^{-tH}\psi_0 = \psi_0 \text{ and } \|\psi_0^2\|_2 = 1$$

In fact, μ is the measure of a reversible diffusion process with invariant measure $d\nu = \psi_0^2\, d\lambda^d$ and stochastic generator H_ν acting in $L^2(\nu)$ as

$$H_\nu f = \frac{1}{\psi_0} H(\psi_0 f) = -\frac{1}{2}\Delta f - \left\langle \frac{\nabla \psi_0}{\psi_0}, \nabla f \right\rangle_{\mathbb{R}^d}.$$

Such processes are called $P(\phi)_1$-processes in [21], although in probability theory they are better known as Itô-diffusions. The transition probabilities for μ are given by

$$\mu(f(\omega_{t+s})|\omega_s = x) = \int Q_t(x,y) f(y)\, d\nu(y), \qquad (2.8)$$

where

$$Q_t(x,y) = \frac{K_t(x,y)}{\psi_0(x)\psi_0(y)} \qquad (2.9)$$

is the transition density of μ with respect to its invariant measure.

$$Q_t(x,y)\; d\nu := \frac{K_t(x,y)}{\psi_0(y)}\; \psi_0(x)\; d\lambda \iff p(y,s,x,t) = k(y,s,x,t)\, \frac{\theta(x,t)}{\theta(y,s)}$$

$$\theta \sim \rho_*^{1/2}$$

Note: fractional Kato class

DEFINITION 2.2. – *A measurable function* $V:\mathbb{R}^d \to \mathbb{R}$ *is said to be in the* Kato class [21] $\mathcal{K}(\mathbb{R}^d)$, *if*

$$\sup_{x\in\mathbb{R}} \int_{\{|x-y|\leqslant 1\}} |V(y)|\,dy < \infty \quad \text{in case } d=1,$$

and

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{\{|x-y|\leqslant r\}} g(x-y)|V(y)|\,dy = 0 \quad \text{in case } d \geqslant 2.$$

Here,

$$g(x) = \begin{cases} -\ln|x| & \text{if } d=2, \\ |x|^{2-d} & \text{if } d \geqslant 3. \end{cases}$$

V *is locally in Kato class, i.e. in* $\mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d)$, *if* $V1_K \in \mathcal{K}(\mathbb{R}^d)$ *for each compact set* $K \subset \mathbb{R}^d$. V *is* Kato decomposable [4] *if*

$$V = V^+ - V^- \quad \text{with} \quad V^- \in \mathcal{K}(\mathbb{R}^d),\ V^+ \in \mathcal{K}_{\mathrm{loc}}(\mathbb{R}^d),$$

where V^+ *is the positive part and* V^- *is the negative part of* V.

Targeted stochasticity idea of I. Eliazar and J. Klafter,
J. Stat. Phys. **111**, 739, (2003)

Lévy-Driven Langevin Systems: Targeted Stochasticity

$$X(dt) = \underbrace{-f(X(t))\,dt}_{\text{Drift}} + \underbrace{L(dt)}_{\text{Driver}}$$

1. **Evolution:** *What is the Fokker–Planck equation governing the evolution of the pdf of the system's state?*

2. **Steady state:** *In steady state, what is the connection between the system's drift function f, driving noise, and stationary pdf?*

3. **Reverse engineering:** *Given a "target" pdf p, can we "tailor design" a drift function f so that the system's stationary pdf would equal the desired "target" pdf p?*

Question: Do we have a guarantee that an invariant density may actually be an asymptotic target ? **Why not by means of semigroups ?**

4. **Boltzmann equilibria:** It is well know that in Wiener-driven Langevin dynamics, i.e., in the Gaussian case (1), the system admits a Boltzmann equilibrium. Namely, the system's stationary pdf equals

$$c \exp\left\{-\frac{2}{\sigma^2}U(x)\right\}, \tag{3}$$

where c is a normalizing constant, σ is the noise amplitude, and U is the external potential. Hence, the following question arises naturally: *Are Boltzmann-type equilibria still attainable when the Lévy driver is non-Gaussian?*

As for the existence of Boltzmann-type equilibria—the following proposition *excludes* their possibility in Lévy driven Langevin systems:

Proposition 5. Boltzmann-type equilibria in the Langevin system (2) are **non**-attainable when the Lévy driver is purely non-Gaussian.

Response to external potentials

Langevin scenario (cf. gradient perturbations)

$$\dot{x} = b(x) + A^{\mu}(t) \Longrightarrow \partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$\partial_t \rho = -\nabla(b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$b = 2D\nabla\Phi$$

$$b(x) = -\lambda \frac{\int |\Delta|^{\mu/2} \rho_*(x)\, dx}{\rho_*(x)}$$

Targeted stochasticity

Lévy-Schrödinger semigroups

$$\hat{H}_\mu \doteq \lambda|\Delta|^{\mu/2} + \mathcal{V}$$

$$\exp(-t\hat{H}_\mu)$$

Schrödinger's boundary data problem

$$\partial_t \theta_* = -\lambda|\Delta|^{\mu/2}\theta_* - \mathcal{V}\theta_*$$

$$\partial_t \theta = \lambda|\Delta|^{\mu/2}\theta + \mathcal{V}\theta$$

$$\theta^*(x,t)\theta(x,t) = \rho(x,t)$$

$$\theta_*(x,t) = \rho(x,t)\,\exp[-\Phi(x)]$$

$$\exp[\Phi(x)] = \rho_*^{1/2}(x)$$

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2}\,\rho_*^{1/2}}{\rho_*^{1/2}}$$

Targeted stochasticity

Transport equation for the pdf looks ugly

$$\partial_t \rho = \theta\partial_t\theta^* = -\lambda(\exp\Phi)|\Delta|^{\mu/2}[\exp(-\Phi)\rho] - \mathcal{V}\cdot\rho$$

„Topologically-” induced jump-type processes and Lévy semigroups

$$\partial_t \rho = -\lambda |\Delta|^{\mu/2} \rho$$

$$(|\Delta|^{\mu/2} f)(x) = -\frac{\Gamma(\mu+1)\sin(\pi\mu/2)}{\pi} \int \frac{f(z)-f(x)}{|z-x|^{1+\mu}}\, dz$$

$$\partial_t \rho(x) = \int [w(x|z)\rho(z) - w(z|x)\rho(x)]\nu_\mu(dz)$$

The jump rate is an even function, $w(x|z) = w(x|z)$

we replace the jump rate $w(x|y) \sim 1/|x-y|^{1+\mu}$

by the expression $w_\phi(x|y) \sim \frac{\exp[\Phi(x)-\Phi(y)]}{|x-y|^{1+\mu}}$

$w_\phi(x|z) \neq w_\phi(z|x)$ → $\partial_t \rho =$?

$$\partial_t \rho = \ ?$$

$$(1/\lambda)\partial_t \rho = |\Delta|_{\Phi}^{\mu/2} f = -\exp(\Phi)\, |\Delta|^{\mu/2}[\exp(-\Phi)\rho] + \rho \exp(-\Phi)|\Delta|^{\mu/2}\exp(\Phi)$$

Whatever potential $\Phi(x)$ has been chosen (up to a normalization factor), then formally $\rho_*(x) = \exp(2\Phi(x))$ is a stationary solution

if for a pre-determined $\rho_* = \exp(2\Phi)$, there exists the semigroup potential $\mathcal{V}$ the dynamics belongs to the semigroup framework.

Rewriting the stationary pdf ρ_* as $\rho_*(x) = (1/Z)\exp(-V_*(x)/k_B T)$ (note **the Gibbs-Boltzmann form** of the pdf !) we get:

$$\partial_t \rho = -\exp(-\kappa V_*/2)\, |\Delta|^{\mu/2}\exp(\kappa V_*/2)\rho + \rho \exp(\kappa V_*/2)|\Delta|^{\mu/2}\exp(-\kappa V_*/2), \ \ \kappa = 1/k_B T.$$

The transport equation has the previous, **semigroup-driven form** !

$$\partial_t \rho = \theta \partial_t \theta^* = -\lambda(\exp\Phi)|\Delta|^{\mu/2}[\exp(-\Phi)\rho] - \mathcal{V}\cdot\rho$$

Quiery: „superdiffusion" ?

Targeted stochasticity for Cauchy driver

$$(|\nabla| f)(x) \;=\; -\frac{1}{\pi}\int \frac{f(z)-f(x)}{|z-x|^2}\,dz$$

Ornstein-Uhlenbeck-Cauchy process

$$\partial_t \rho = -\lambda |\nabla| \rho + \nabla[(\gamma x)\rho]$$

$$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

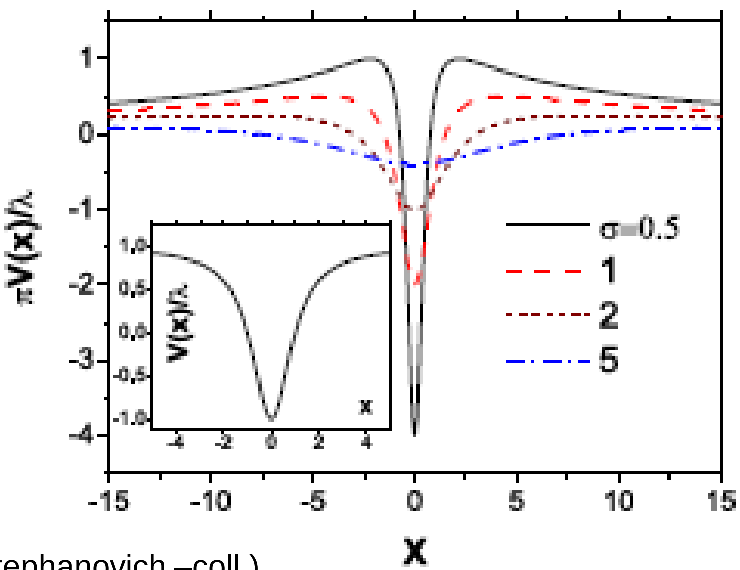
$$\sigma = \frac{\lambda}{\gamma},$$

Invariant density vs semigroup potential

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2}\,\rho_*^{1/2}}{\rho_*^{1/2}}$$

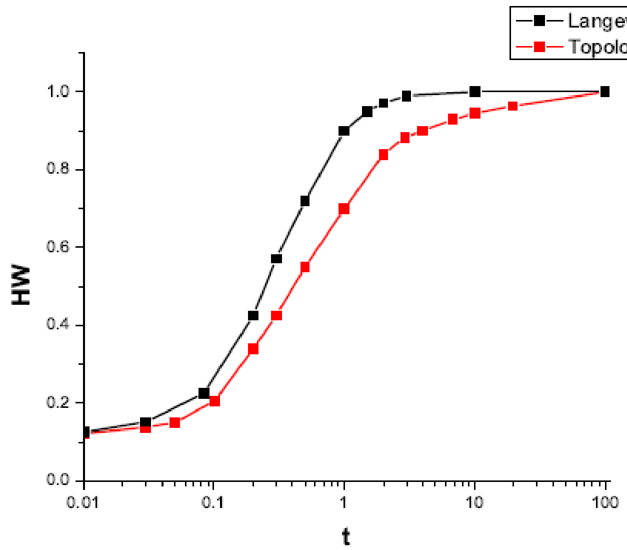
$$\mathcal{V}(x) = \frac{\lambda}{\pi}\left[-\frac{2}{\sqrt{a}} + \frac{x}{a}\ln\frac{\sqrt{a}+x}{\sqrt{a}-x}\right]$$

$$a = \sigma^2 + x^2$$



(V. Stephanovich –coll.)

Targeted stochasticity in the **time domain**



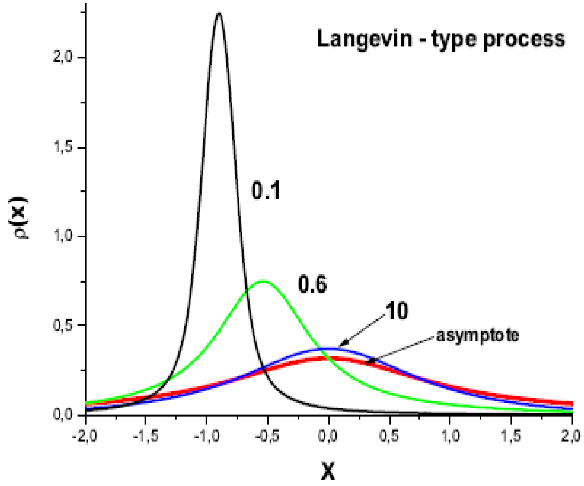


FIG. 1: Temporal behavior of the half-maximum width (HW): for the OUC process in Langevin-driven and semigroup-driven (topological) processes. Motions begin from common initial data $\rho(x, t = 0) = \delta(x)$ and end up at a common pdf (20) for $\sigma = 1$.

FIG. 2: Time evolution of Langevin-driven pdf $\rho_L(x, t)$ beginning from the initial data $\rho_L(x, t = 0) = \delta(x + 1)$ and ending at the pdf (20) (shown as "asymptote" in the figure) for $\sigma = 1$. Figures near curves correspond to t values.

Dynamics in the OUC process with: $$\rho_*(x) = \frac{\sigma}{\pi(\sigma^2 + x^2)}$$

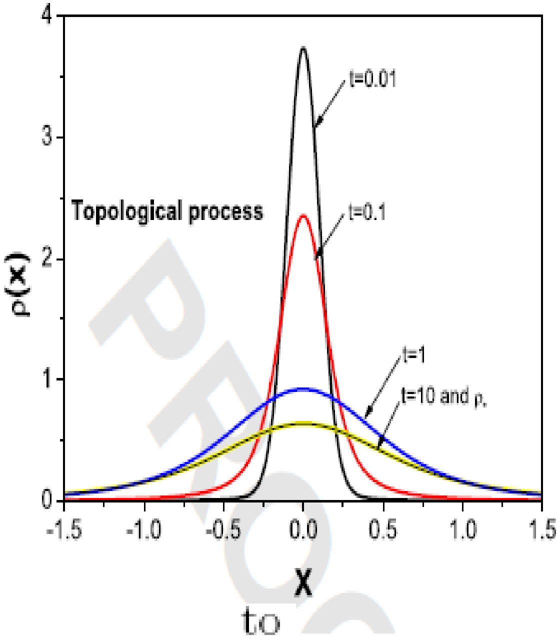
Targeted stochasticity in the **time domain (confined noise)**

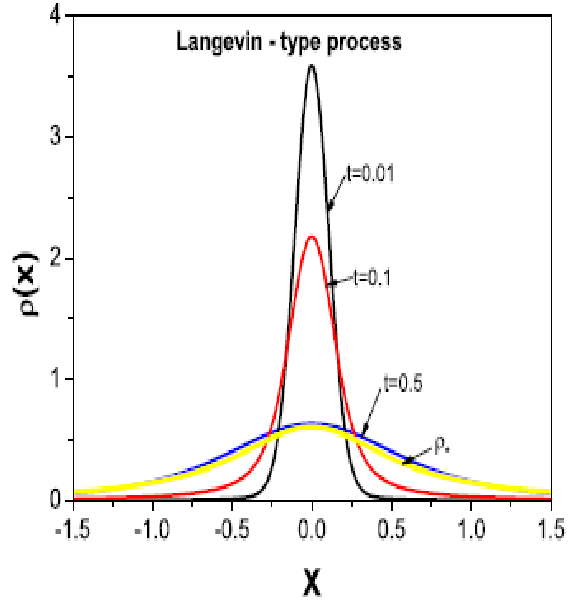
Langevin drift $$b(x) = -\frac{\gamma x}{8}(x^2+3)$$

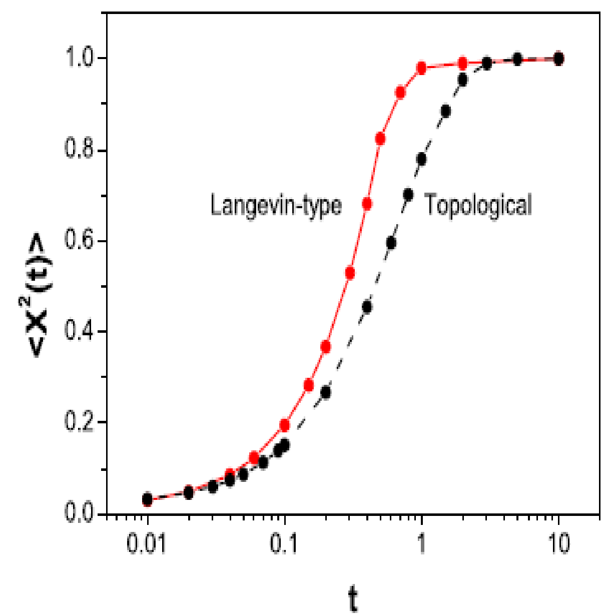
Invariant density

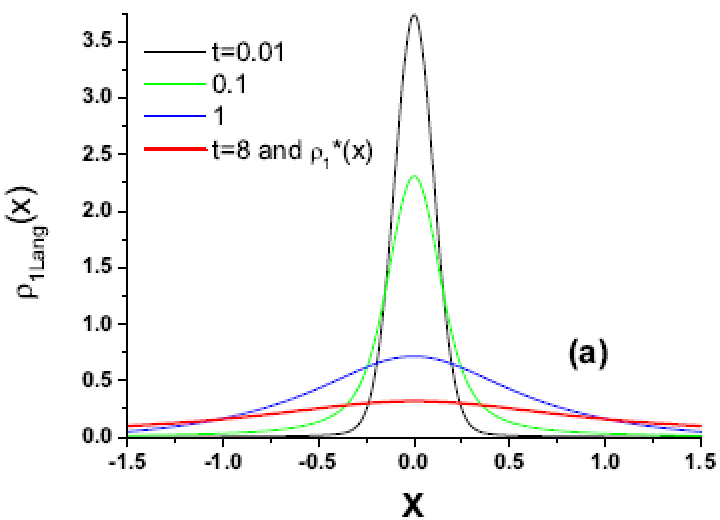
$$\rho_*(x) = \frac{2}{\pi}\frac{1}{(1+x^2)^2}$$

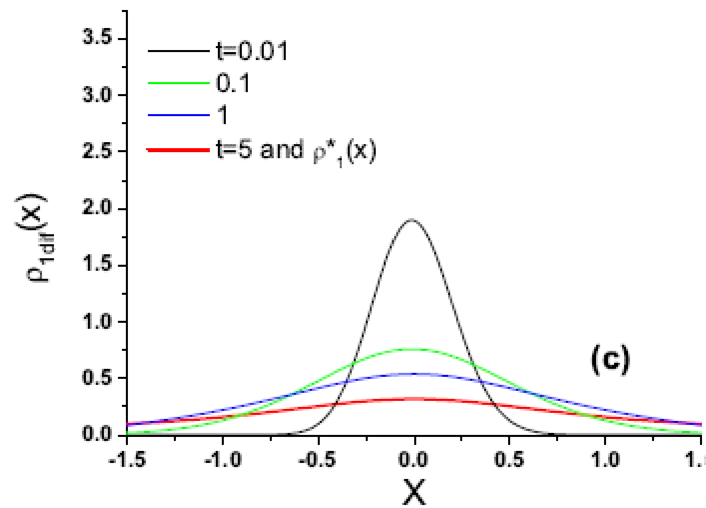
Semigroup potential $$\mathcal{V}(x) := \lambda \frac{x^2-1}{x^2+1}$$











Diffusive scenario !

$$D = 1;\ b = b_{diff} = \nabla \ln \rho_*$$

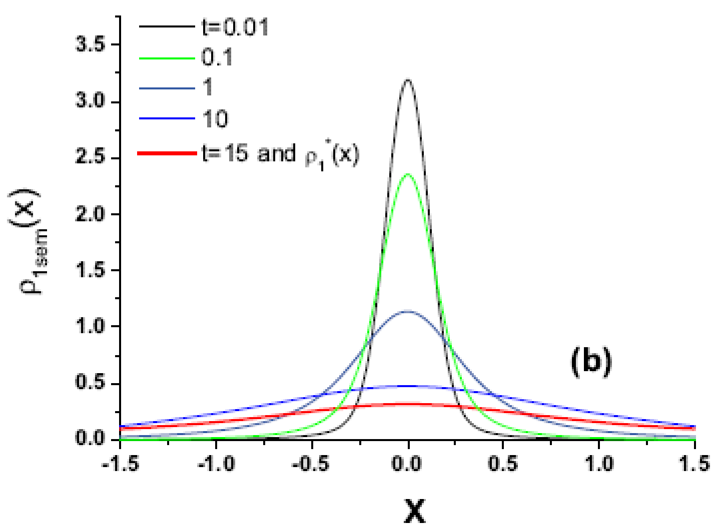


FIG. 2: Time evolution of pdf's $\rho(x,t)$ for the Cauchy-Langevin dynamics (panel (a)), Cauchy-semigroup-induced evolution (panel (b)) and the Wiener-Langevin process (panel (c)). The common target pdf is the Cauchy density, while the initial $t = 0$ pdf is set to be a Gaussian with height 25 and half-width $\sim 10^{-3}$. The first depicted stage of evolution corresponds to $t = 0.01$. The time rate hierarchy seems to be set: diffusion being fastest, next Lévy-Langevin and semigroup-driven evolutions being slower than previous two. However the outcome is not universal, as will show our further discussion.

„superdiffusion" ? Not quite…

Cauchy semigroup: **false** Gibbs- Boltzmann asymptotics

$$\partial_t \rho = -\exp(-\kappa V_*/2)\,|\Delta|^{\mu/2}\exp(\kappa V_*/2)\rho + \rho\exp(\kappa V_*/2)|\Delta|^{\mu/2}\exp(-\kappa V_*/2)$$

$\rho_*^{1/2}(x) = \exp(\Phi(x)) = (1/\sqrt{Z})\exp(-V_*(x)/2k_B T)$ $\kappa = 1/k_B T$

We **scale away** dimensional units and consider **typical Gibbs-Boltzmann** forms of $\rho_*^{1/2}(x)$: with $V_*(x) \equiv \Phi(x) = x^4 - 2x^2 + 1$ and $\Phi \equiv V_*(x) = x^2$

Hyper-confinement

Compare V_* with $\mathcal{V}$

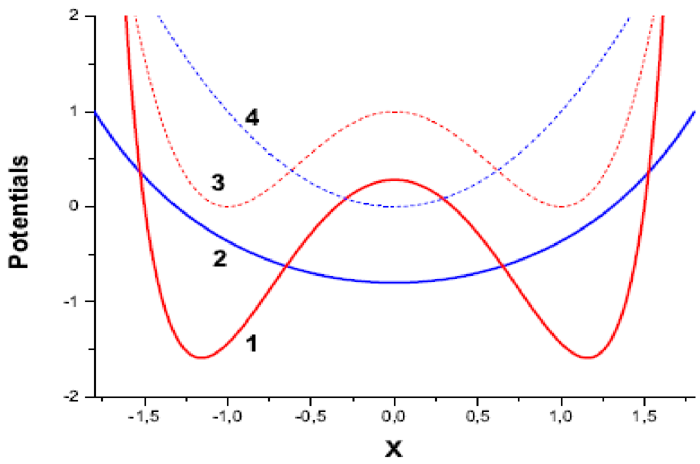


FIG. 4: The coordinate dependence of the semigroup potential $\mathcal{V}(x)$ (curves 1 and 2), corresponding to $V_*(x) = x^4 - 2x^2 + 1$ (curve 3) and $V_*(x) = x^2$ (curve 4), respectively. Curves 3 and 4 are shown for a comparison with, strikingly similar in shape, semigroup potential curves 1 and 2

Direct **semigroup** inference: Cauchy oscillator

$$\hat{H}_{1/2} \equiv \lambda|\nabla| + \left(\frac{\kappa}{2}x^2 - \mathcal{V}_0\right)$$

$$\hat{H} = -D\Delta + \left(\frac{\gamma^2 x^2}{4D} - \frac{\gamma}{2}\right)$$

direct reconstruction route:

$$\left(\frac{\kappa}{2}x^2 - \mathcal{V}_0\right)\rho_*^{1/2} = -\lambda\,|\nabla|\,\rho_*^{1/2}$$

$\tilde{f}(p)$ the Fourier transform of $f = \rho_*^{1/2}(x)$

$$-\frac{\kappa}{2}\Delta_p\tilde{f} + \gamma|p|\tilde{f} = \mathcal{V}_0\tilde{f}$$

$k = (p - \sigma)/\zeta$ $\quad \psi(k) = \tilde{f}(p)$ $\quad \sigma = \mathcal{V}_0/\gamma$ $\quad \zeta = (\kappa/2\gamma)^{1/3}$

$$\frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$$

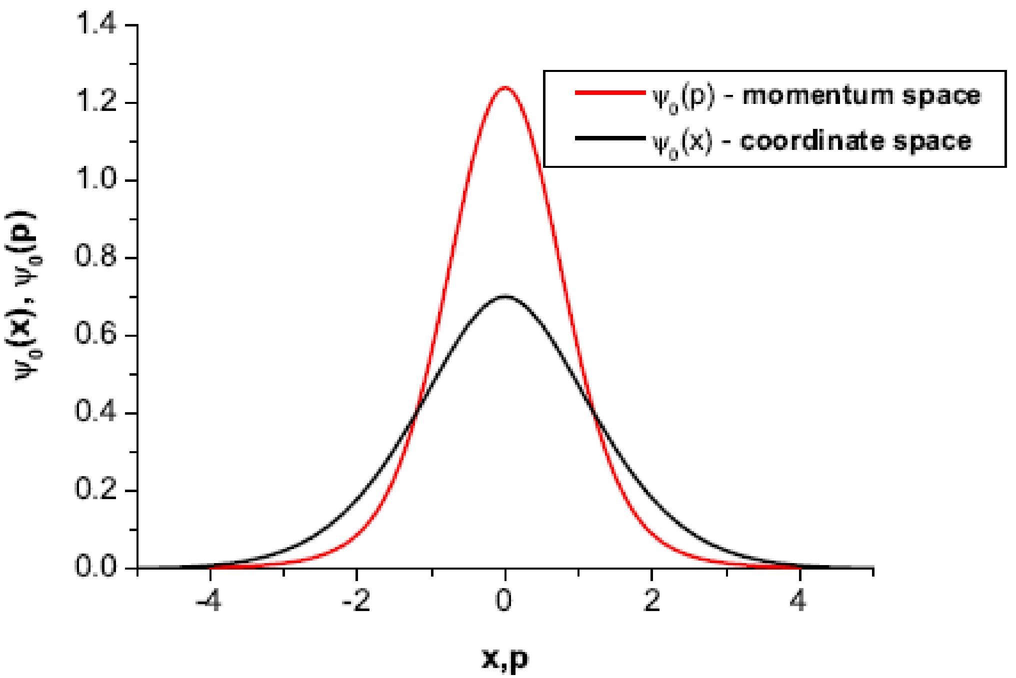
A unique normalized ground state function of $$\frac{d^2\psi(k)}{dk^2} = |k|\psi(k)$$

is composed of two Airy pieces

that are glued together at the first zero y_0 of the Airy function derivative:

$$\psi_0(k) = A_0 \begin{cases} \mathrm{Ai}(-y_0 + k), & k > 0 \\ \mathrm{Ai}(-y_0 - k), & k < 0, \end{cases}$$

$$A_0 = \left[\mathrm{Ai}(-y_0)\sqrt{2y_0}\right]^{-1}, \quad y_0 \approx 1.01879297$$



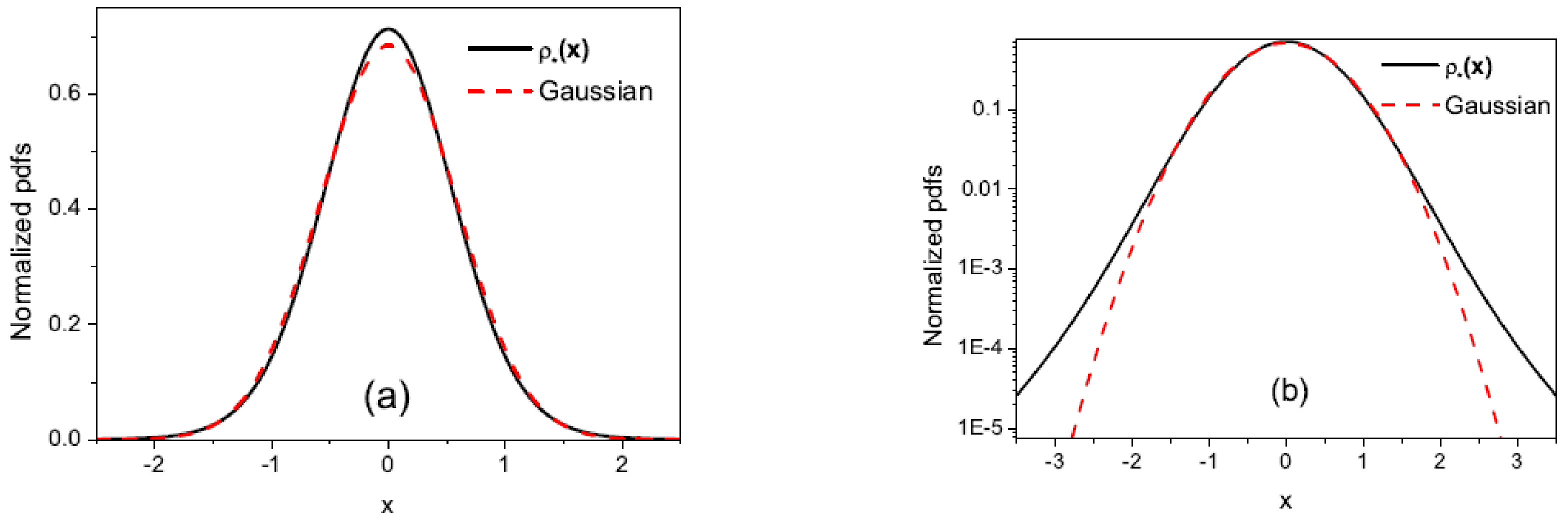


FIG. 7: Normalized invariant pdf (30) (full line) for the quadratic semigroup potential. The Gaussian function, centered at $x = 0$ and with the same variance $\sigma^2 = 0.339598$ is shown for comparison. Panel (a) shows functions in linear scale, while panel (b) shows them in logarithmic scale to better visualize their different behavior.

$$\psi_0(x) = \frac{A_0}{\pi} \int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t + y_0) dt = \rho_*^{1/2}(x)$$

Reverse engineering for the Cauchy oscillator ground state pdf

For a given ρ_* the definition of a drift function $b(x)$ (we put either $\lambda = 1$ or define $b \to b/\lambda$) is:

$$b(x) = -\frac{1}{\rho_*(x)} \int [|\nabla|\rho_*(x)]dx \equiv$$

$$\frac{1}{\pi \rho_*(x)} \int dx \int_{-\infty}^{\infty} \frac{\rho_*(x+y) - \rho_*(x)}{y^2} dy \, .$$

Inserting $\rho_*(x)$, Eq. (30), we get

Lévy- Langevin drift

$$b(x) = -\frac{\int_{-y_0}^{\infty} \mathrm{Ai}(t) \sin x(t+y_0) dt}{\int_{-y_0}^{\infty} \mathrm{Ai}(t) \cos x(t+y_0) dt} .$$

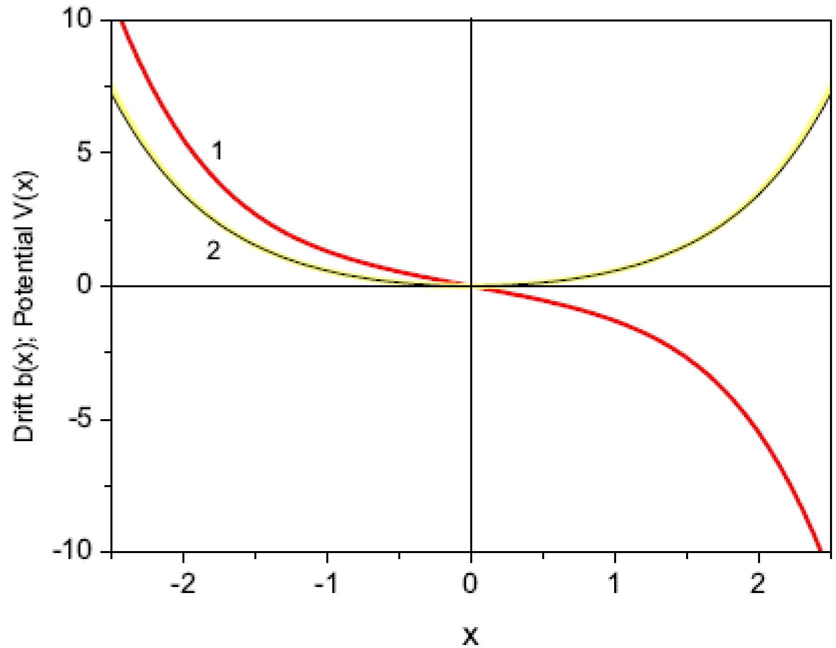


FIG. 8: Langevin - type drift $b(x)$ (curve 1) and its (force) potential $V(x)$ (curve 2), that give rise to an invariant density (30).

Confinement hierarchy - case study

$$\rho_*(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha - 1/2))} \frac{1}{(1+x^2)^\alpha} \qquad \alpha > 1/2$$

Semigroup reconstruction

$$\mathcal{V} = -\lambda \frac{|\Delta|^{\mu/2} \rho_*^{1/2}}{\rho_*^{1/2}}$$

$$\partial_t \Psi = -\lambda |\Delta|^{\mu/2} \Psi - \mathcal{V} \Psi$$

$$\rho(x,t) = \Psi(x,t) \rho_*^{1/2}(x)$$

Langevin drift reconstruction

$$b(x) = -\frac{\gamma}{\rho_*(x)} \int (|\nabla| \rho_*)(x)\, dx$$

$$\partial_t \rho = -\nabla (b \cdot \rho) - \lambda |\Delta|^{\mu/2} \rho$$

$$\partial_t \rho_* = 0 = -\nabla (b\, \rho_*) - \gamma |\nabla| \rho_*$$

That was about **jump-type** processes.
What about **diffusion-type alternative,** with the Gibbs-Boltzmann ansatz, like e.g.

$$\rho_*(x) = C \exp(-\lambda V(x)).$$ and $$b \sim -\nabla V$$

Trial potential: V(x) ~ $\ln(1 + x^2)$

Min/max information entropy principle

To have a better insight into the extremum principles at work, let us recall the standard maximum (information/Shannon) entropy principle: consider $[a, b] \in R$, assume that everything you know about the a priori unknown probability measure are (possibly) its moments

$$\int_a^b x^k \rho(x) \mathrm{d}x = m_k \tag{51}$$

with $k = 0, 1, \ldots, M$ and $m_0 = 1$ — the normalization condition.

We look for densities that maximize the Shannon entropy of a continuous probability distribution (now we encounter a functional of a concave function):

$$\mathcal{S}[\rho] = -\int_a^b \rho \ln \rho \mathrm{d}x \tag{52}$$

under the constraint of M fixed moments [5, 17].

The extremum of a functional

$$\tilde{\mathcal{S}} = -\int_a^b \rho \ln \rho \mathrm{d}x + \sum_0^M \lambda_k \left(\int_a^b x^k \rho \mathrm{d}x - m_k \right) \tag{53}$$

(a concavity property of $\mathcal{S}$ needs to be remembered) sets the functional form of ρ which maximizes the entropy

$$\rho_*(x) = C \exp\left(-\sum_a^b \lambda_k x^k \right), \tag{54}$$

where $C = \exp(-\lambda_0 - 1)$ is the normalization constant and λ_k's are fixed by identities

$$\int_a^b x^k \exp\left(-\sum_a^b \lambda^k x^k \right) \mathrm{d}x = m_k. \tag{55}$$

If there is a unique solution in terms of $\lambda_1, \ldots, \lambda_M$, we say that an entropy maximizing (under the m_k "circumstances") density does exist.

For reference, let us reproduce some pieces of a standard wisdom:

(i) If a and b are finite, there exists a unique maximum entropy density.

(ii) In R^+, e.g. $[0, +\infty)$, a maximizing density exists if $m_1^2 \leq m_2 \leq 2m_1^2$.

Notes: if there is no constraint, there is no maximizing density; if only the mean $m_1 = 1/\alpha$ is given, we get the exponential one: $\rho_*(x) = \alpha \exp(-\alpha x)$; for the Gaussian on R^+, like e.g. $\rho(r) = (2/\sqrt{\pi}) \exp(-r^2)$, we have $\mathcal{S}(\rho) = (\ln \pi + 1)/2$, which is a maximum of the Shannon entropy under the moment constraints $m_1 = \langle r \rangle = 1/\sqrt{\pi}$ and $m_2 = \langle r^2 \rangle = 1/2$; for another Gaussian on R^+, $P_0(s) = (2/\pi) \exp(-s^2/\pi)$, we have $m_1 = \langle s \rangle = 1$, $m_2 = \pi/2$ and $\mathcal{S}(\rho) = [\ln(\pi^2/4) + 1]/2$.

(iii) In R, with no moment prescribed, or given the mean only, there is no maximum entropy density.

Notes: if m_1 and m_2 are given, the maximum entropy distribution is the normal (Gaussian) one, with the variance $\sigma^2 = m_2 - m_1^2$, i.e. $\rho(x) = \exp(-(x - m_1)^2/2\sigma^2)/(\sqrt{2\pi}\sigma)$ and the Shannon entropy value is $\mathcal{S}(\rho) = \ln(2\pi e \sigma^2)/2$. That is to be compared with the previous outcome, Eq. (28), for the Gaussian on R^+.

Entropy extremum principle

Fix a priori the value of

$$\mathcal{U} = \int_{-\infty}^{\infty} \ln(1+x^2)\, \rho(x) dx \quad \boxed{= \zeta}$$

(x carries **no** dimension, $x \equiv x/x_0$)

Extremize an obvious Helmholtz free energy analog ($F = U - TS$)

$$\mathcal{F} = \alpha \, \langle \ln(1+x^2) \rangle - \mathcal{S}(\rho)$$

$\mathcal{S}(\rho) = -\langle \ln \rho \rangle$, while α is a Lagrange multiplier to be explicitly inferred in the variational procedure. $\zeta \longleftrightarrow \alpha$

$$\delta \mathcal{F}(\rho)/\delta \rho = 0$$

$$\downarrow$$

$$\rho_\alpha(x) = (1/Z_\alpha)\,(1+x^2)^{-\alpha}$$

provided the normalization factor $Z_\alpha = \int_{-\infty}^{\infty} (1+x^2)^{-\alpha}\, dx$ exists.

To identify the value of the Lagrange multiplier α, we need

$$\mathcal{U}_\alpha = \frac{\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha - 1/2)} \int_{-\infty}^{\infty} \frac{\ln(1+x^2)}{(1+x^2)^\alpha} dx.$$

With an explicit expression for Cauchy family pdfs in hands, we readily evaluate Shannon entropy

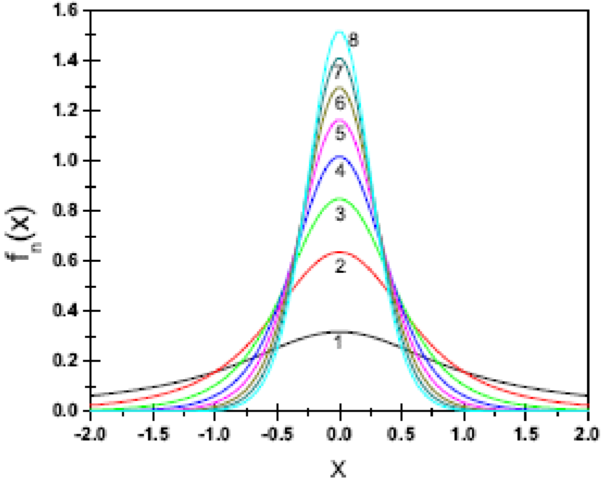
$$\mathcal{S}_\alpha = -\int_{-\infty}^{\infty} \rho_\alpha(x) \ln \rho_\alpha(x) dx = \ln Z_\alpha + \alpha \mathcal{U}_\alpha.$$

and Helmholtz free energy analog

$$\mathcal{F}_\alpha = \alpha \mathcal{U}_\alpha - \mathcal{S}_\alpha \equiv -\ln Z_\alpha$$

In view of the divergence of Z_α, both the Shannon entropy and the Helmholtz free energy (likewise $\mathcal{U}_\alpha$) cease to exist at $\alpha = 1/2$.

Given the internal energy value, we can read out the corresponding Lagrange multiplier value from the figure



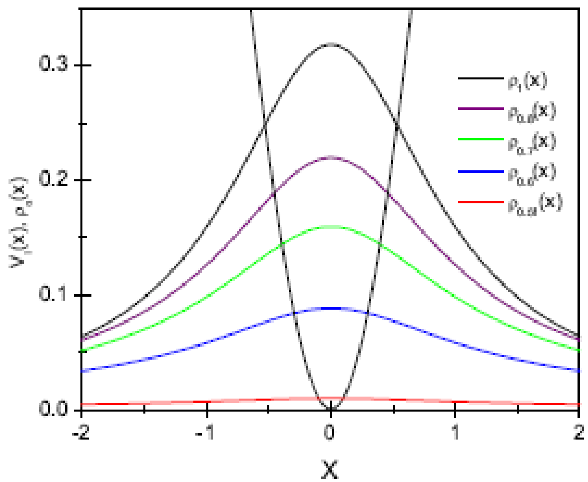


FIG. 1: Logarithmic potential $\ln(1+x^2)$ against $\rho_\alpha(x)$, with $1/2 < \alpha \leq 1$ and for some integer α.

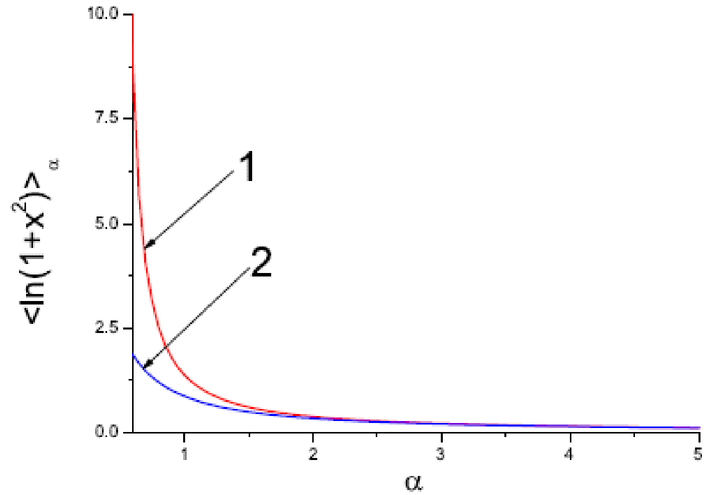


FIG. 2: $\mathcal{U}_\alpha$ (curve 1) and its asymptotic expansion at large α (curve 2).

Note: the number of moments grows from none at all to infinity

Thermalization argument (limitations upon states of thermal equilibrium)

$$\partial_t \rho = D\Delta\rho - \nabla[-\rho\nabla V(x)/m\gamma]$$

$$D = k_B T/m\gamma \qquad \partial_t \rho = \nabla[\rho\nabla\Psi]/(m\gamma)$$

Consider:

$$\Psi = V + k_B T \ln \rho \tag{21}$$

whose mean value is indeed the Helmholtz free energy of random motion

$$F \equiv \langle\Psi\rangle = U - TS. \tag{22}$$

Here the (Gibbs) entropy reads $S = k_B\mathcal{S}$, while an internal energy is $U = \langle V\rangle$. In view of assumed boundary restrictions at spatial infinities, we have $\dot{F} = -(m\gamma)\left\langle v^2\right\rangle \leq 0$, where $v = -\nabla\Psi/(m\gamma)$. Hence, F decreases as a function of time towards its minimum F_*, or remains constant.

At equilibrium:

$$\Psi_* = V + k_B T \ln \rho_* \Longrightarrow \langle \Psi_* \rangle = -k_B T \ln Z \equiv F_*$$

To be compared with the previous (note dimensional issues we bypass !)

$$\mathcal{F}_\alpha = \alpha \mathcal{U}_\alpha - \mathcal{S}_\alpha \equiv - \ln Z_\alpha$$

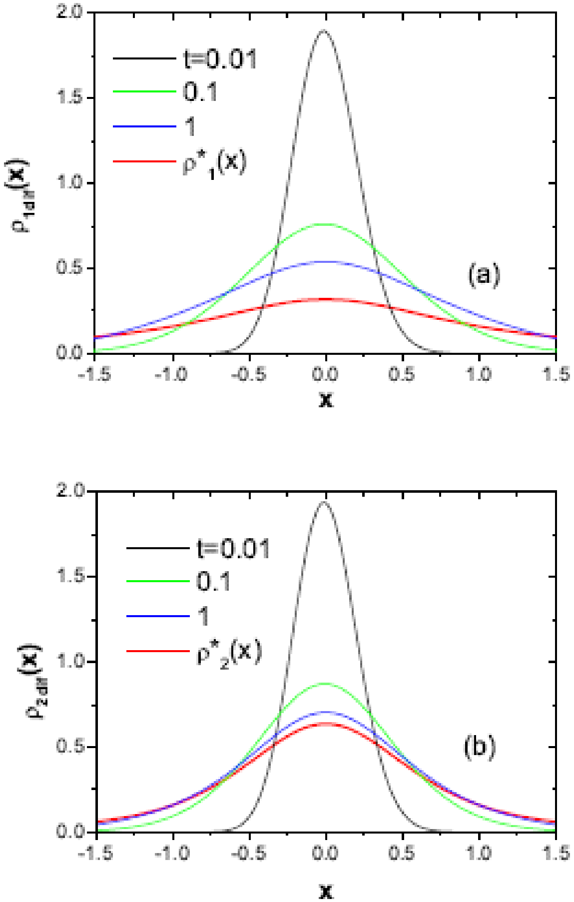
Once we choose $V(x) = \epsilon_0 \ln(1 + x^2)$ and

(at fixed energy scale ϵ_0) $\alpha = \epsilon_0/(k_B T)$

$$\rho_*(x) = \frac{\Gamma(\alpha)}{\sqrt{\pi}\Gamma(\alpha - 1/2))} \frac{1}{(1 + x^2)^\alpha}$$

$\alpha \to \infty$ corresponds to $T \to 0$
Since $1/2 < \alpha$, the temperature scale, within which our system may at all be set at thermal equilibrium, is bounded: $0 < k_B T < 2\epsilon_0$.

$\alpha = 1$ i.e. $k_B T = \epsilon_0$ corresponds to Cauchy density.



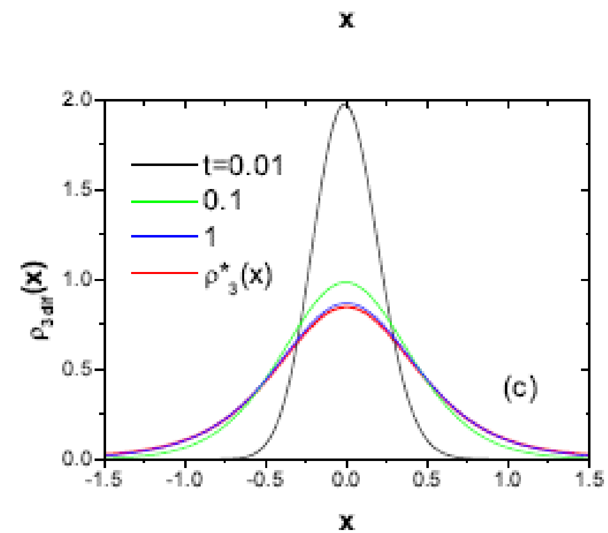
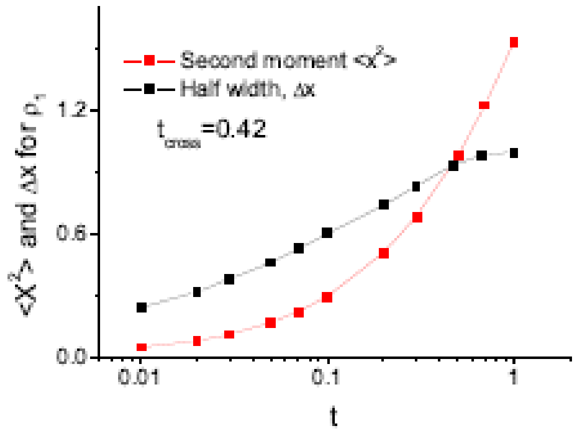
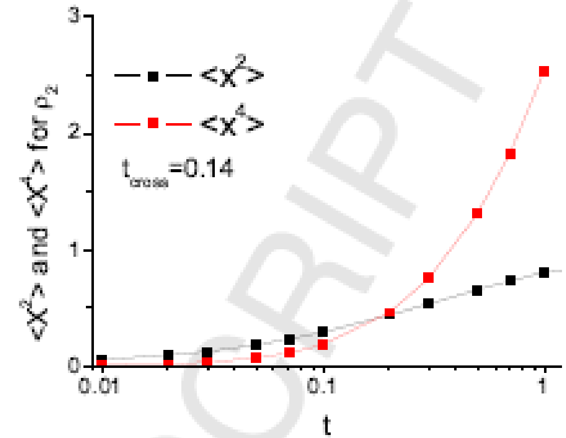


FIG. 6: Time evolution of pdf's $\rho(x,t)$ for Smoluchowski processes in logarithmic potential $\ln(1+x^2)$. The initial $(t=0)$ pdf is set to be a Gaussian with height 25 and half-width $\sim 10^{-3}$. The first depicted stage of evolution corresponds to $t = 0.01$. Target pdfs are the members of Cauchy family for $\alpha = 1, 2, 3$ respectively.

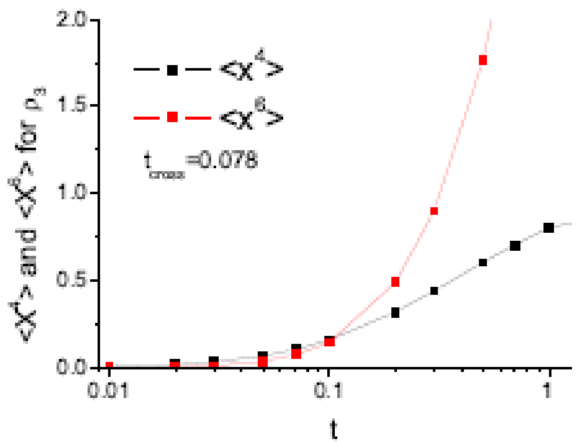
Transient dynamics in Brownian motion



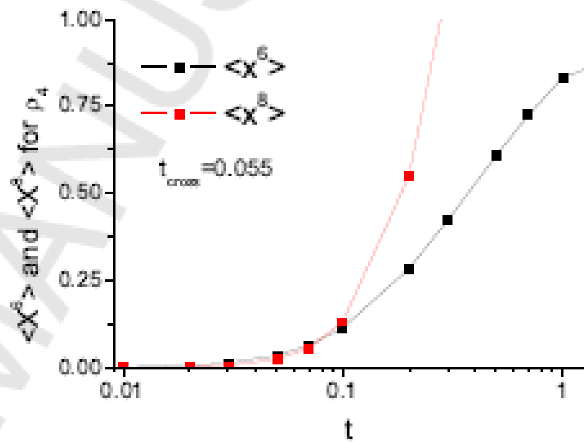
(a) Half-width and $< X^2 >$ (divergent) for ρ_{*1}



(b) $< X^2 >$ and $< X^4 >$ (divergent) for ρ_{*2}



(c) $< X^4 >$ and $< X^6 >$ (divergent) for ρ_{*3}



(d) $< X^6 >$ and $< X^8 >$ (divergent) for ρ_{*4}

The time evolution of the last convergent and first divergent moment for pdfs ρ_{*1} - ρ_{*4}, shown on panels (a) - (d)

Exponential families of pdfs

$$\partial_t \rho = \Delta\rho - \nabla(b\,\rho)$$

(we have set D =1 in $\partial_t \rho = D\Delta\rho - \nabla(b\cdot\rho)$)

$\rho_0(x)$ → $\rho_*(x)$

$$\rho_*(x) = \exp[\ln \rho_*(x)] = A\,\exp(-V(x))$$
$$\Downarrow$$
$$\rho_\alpha = A_\alpha\,\exp(-\alpha\,V(x))$$

$$\alpha = \epsilon_0/(k_B T)$$

Trial ansatz: (i) $V(x) = x^2$

(ii) $V(x) = \ln(1+x^2)$

Exponential Gauss family

$$\rho_*(x) = \sqrt{\frac{1}{\pi}} \exp(-x^2) = A \exp[-V(x)]$$

$$V(x) = x^2$$

$$\rho_\alpha(x) = \sqrt{\frac{\alpha}{\pi}} \exp(-\alpha x^2)$$

$$\alpha = \epsilon_0/(k_B T)$$

$$b_\alpha = b_{\alpha,diff} = \nabla \ln \rho_\alpha = -\alpha \nabla V(x) = -2\alpha x$$

Exponential Cauchy family

$$\rho_*(x) = \frac{1}{\pi}\frac{1}{1+x^2} = \exp(-\ln\pi)\ \exp[-\ln(1+x^2)] = A\ \exp[-V(x)]$$

$$V(x) = \ln(1+x^2)$$

$$\rho_\alpha = \frac{A_\alpha}{(1+x^2)^\alpha} \equiv \exp(\ln\rho_\alpha)$$

$$\rho_\alpha = A_\alpha\ \exp[-\alpha V(x)]$$

$$\alpha\ =\ \epsilon_0/(k_B T)$$

$$\ln\rho_\alpha = \ln A_\alpha - \alpha\ \ln(1+x^2)$$

$$b_\alpha = b_{\alpha,diff} = \nabla\ln\rho_\alpha = -\alpha\nabla V(x) = -\alpha\frac{2x}{1+x^2}$$

Exponential familiy for any $\rho_*(x)$

Select $\rho_*(x)$ for which an exponential family is to come out

$$\rho_*(x) = \exp[\ln \rho_*(x)] = A \exp(-V(x))$$
$$\Downarrow$$
$$\rho_\alpha = A_\alpha \exp(-\alpha V(x))$$

$$b_\alpha = b_{\alpha,diff} = \nabla \ln \rho_\alpha$$

Fix a priori the value of

$$\mathcal{U} = \int V(x)\rho(x)dx = \langle V \rangle = \zeta$$

Extremize with respect to ρ

$$\mathcal{F} = \alpha \langle V \rangle - \mathcal{S}(\rho)$$

$$\mathcal{S}(\rho) = -\langle \ln \rho \rangle$$

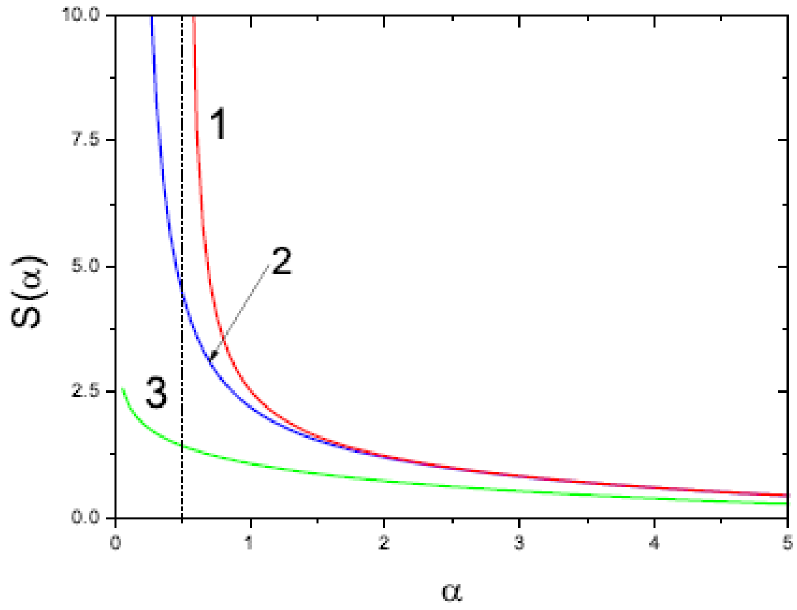
α is a Lagrange multiplier whose concrete value (to be inferred) depends on ζ: $\zeta \longleftrightarrow \alpha$.

Cauchy vs Gauss: α interpolation

$$\rho_\alpha(x) = \frac{1}{Z_\alpha}(1+x^2)^{-\alpha}$$

$$\rho_\alpha^G(x) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2} \equiv \frac{1}{Z_\alpha^G}\, e^{-\alpha x^2}$$

$$\rho_\alpha(x) \approx \left[\sqrt{\frac{\alpha}{\pi}} - \frac{3}{8\sqrt{\pi\alpha}} + \ldots\right] \exp\left(-\alpha x^2 + \alpha\frac{x^4}{2} + \ldots\right)$$



$\alpha > 5$ sets

a fairly good approximation

Fig. 3: (Color online) $\mathcal{S}_\alpha$ for Cauchy family (curve 1) and its asymptotic expansion at large α (curve 2). $\mathcal{S}_\alpha^G$ for Gaussian family is also shown (curve 3).

Link with Tsallis entropies and pdfs

$$S_q[\rho] = \frac{k_B}{q-1}\left(1 - \int d(x/\sigma)[\sigma\rho(x)]^q\right)$$

$$\langle x^2\rangle_q = \int d(x/\sigma)x^2\,[\sigma\rho(x)]^q = \sigma^2,$$

$$\rho_q(x) = \frac{1}{Z_q}[1-\beta(1-q)x^2]^{1/1-q}$$

$$Z_q = \left[\frac{\beta(q-1)}{\pi}\right]^{1/2}\frac{\Gamma(1/(q-1))}{\Gamma((3-q)/2(q-1))}$$

we need $1 \leq q < 3$ to secure the convergence of the normalization integral

$$V(x) = \frac{1}{\beta(q-1)}\ln[1+\beta(q-1)x^2]$$

redefine the constants involved

$$\beta = \frac{\alpha}{2D}, \qquad q = 1 + \frac{2D}{\sigma^2\alpha}$$

$$\Rightarrow \quad V(x) = \sigma^2\ln[1+(x/\sigma)^2]$$

$$\rho_q(x) = \frac{1}{Z_q}[1+(x/\sigma)^2]^{-\beta\sigma^2}$$

β plays the role of $1/k_BT$

What was all that about ?

Heavy-tailed targets and (ab)normal asymptotics in diffusive motion

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We show that under suitable confinement conditions, the ordinary Fokker-Planck equation may generate non-Gaussian heavy-tailed probability density functions (pdfs) (like e.g. Cauchy or more general Lévy stable distributions) in its long time asymptotics. In fact, all heavy-tailed pdfs known in the literature can be obtained this way. For the underlying diffusion-type processes, our main focus is on their transient regimes and specifically the crossover features, when initially infinite number of the pdf moments drops down to a few or none at all. The time-dependence of the variance (if in existence), $\sim t^{\gamma}$ with $0 < \gamma < 2$, in principle may be interpreted as a signature of sub-, normal or super-diffusive behavior under confining conditions; the exponent γ is generically well defined in substantial periods of time. However, there is no indication of any universal time rate hierarchy, due to a proper choice of the driver and/or external potential.

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