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QUANTUM MEANING OF CLASSICAL (FIELD) THEORY FOR FERMI SYSTEMS

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We give a path integral reconstruction of two quantum problems: the Fermi oscillator and the chiral invariant Gross–Neveu model. Integrations are carried out with respect to genuine (i.e. c-number, non-Grassmann) paths, so that one is able to identify explicitly contributions of the c-number classical models to the transition amplitudes of their quantized Fermi partners. Stationarity "points" of the respective actions are verified to coincide with the classical trajectories: of the classical oscillator and of the classical chiral invariant Gross–Neveu field.

1. Motivation

Simple Fermi systems if considered on a lattice usually allow a reconstruction as the interacting (many-body) lattices of spins $\frac{1}{2}$. A converse mapping is realizable as well, and in $1+1$ dimensions it amounts to making the Jordan–Wigner transformation from spins $\frac{1}{2}$ to fermions or inversely. Since for spin-$\frac{1}{2}$ lattices it is of some interest (computation of the partition function) to build a complete path integral representation of the theory, one is tempted to invent at least a rough "path" notion in the appropriate classical phase space of the single spinning system or of the collection of them. In this connection we find it instructive to catalogue the up-to-date variety of approaches.

(i) The most popular, say pragmatic, route is to construct path integrals not for lattice spins $\frac{1}{2}$ themselves, but rather for the related Fermi systems, and then in terms of Grassmann algebra variables. Even if one starts from the lattice spin-$\frac{1}{2}$ system, take an Ising model as example, one finds reasonable a transformation to Fermi variables, which is followed by the formal path integral representation of such a "fermionized" system in terms of anticommuting (Grassmann) objects [1]. The computational reliability of this method is rather obvious.

(ii) On the other hand the semiclassical quantization procedure for the continuous Heisenberg system [2] resulted in the introduction of the spin path integral with respect to genuine (i.e. c-number, non-Grassmann) paths in the phase space of the classical spin system.

The genuine c-number path notion is also inherent in the approach of [3–6] based on the SU(2) phase variables, and making use [4–6] of spin-coherent states,

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these being well known in the many-body physics. The general method of [6] to construct measures for spin-variable path integrals can be immediately adopted to either a single spin-$\frac{1}{2}$ or to the many-body spin-$\frac{1}{2}$ problem (which in turn has an equivalent Fermi realization to which the case (i) applies).

(iii) Attempts to introduce probabilistic ideas (and in this connection genuine probabilistic measures) to the study of Fermi fields [7, 8] start from an appealing assumption. Consider the classical harmonic oscillator problem, view its equations of motion as stochastic differential equations (to realize, furthermore, Nelson's quantization procedure), and then add the information that one is considering the two-level Fermi system instead of the ordinary quantum Bose oscillator. It results in specifying the class of stochastic processes in which solutions of the would-be classical oscillator equations of motion: $\dot{Q}=P$, $\dot{P}=-Q$ are to be found. The underlying processes are Markov processes $q(t)$ with values in Z_2, which demonstrates that except for the form of the equations of motion, the "classical" paths of the (Fermi) system are completely unrelated to the Bose oscillator paths.

An analogous line is followed in [9], though in a different, Poisson process, framework.

(iv) In papers [10, 11] which take inspiration from a much earlier [12] attempt to describe Fermi fields in terms of c-number path integrals, the starting point is an embedding of representations of the CAR (Fermi) algebras in those of the CCR (Bose) algebras. This rather special realization of the so called "bosonization" recipes was proposed in [13], see also [14], and allows for the use of the coherent state machinery well developed in the Bose case. In fact one is able to relate the classical phase-space notion known to be applicable to Bose systems with the classical phase-space notion for spin-$\frac{1}{2}$ or Fermi systems, thus resulting in the genuine c-number path integral representation.

Below we shall give an explicit description of how: (i) the classical oscillator problem can be related to the Fermi oscillator problem, (ii) the c-number spinor field theory model known as the chiral invariant Gross–Neveu model [15] can be related to its well-known asymptotically free quantized Fermi partner. In both of these cases we derive the c-number path integral representation of the quantum Fermi models, which explicitly recover the "quantum meaning of classical (field) theory" [16] for spinor systems.

2. Fermi oscillator

Our starting point is the simple Fermi oscillator defined by the lagrangian

$$\hat{L}=ia^*(t)\dot{a}(t)-\omega a^*(t)a(t)=a^*(t)\left[i\frac{\mathrm{d}}{\mathrm{d}t}-\omega\right]a(t)\,, \tag{2.1}$$

which is completely described in terms of the CAR algebra generators (equal time variables omitted for simplicity):

$$[a, a^*]_+ = 1_F , \quad a^*|0) = |1) , \quad a|0) = 0, \quad a^{*2} = 0 = a^2 . \tag{2.2}$$

Because we work with a Fock representation of the CAR algebra, there is an apparent embedding of it in the Fock representation of the CCR algebra (take the Schrödinger representation as an example)

$$[b, b^*]_- = 1_B , \quad b|0) = 0 , \quad \frac{1}{\sqrt{n!}} b^{*n}|0) = |n) , \tag{2.3}$$

provided we define

$$a^* = b^* :\exp(-b^*b): , \quad a = :\exp(-b^*b): b . \tag{2.4}$$

These operators though defined in the whole representation (Hilbert) space h for (2.3) act non-trivially on a proper subspace h_F of h only, with h_F being spanned by vectors $|0)$ and $|1)$ of the initial Bose problem. We have

$$[a, a^*]_+ = :\exp(-b^*b): + b^* :\exp(-b^*b): b = 1_F, \tag{2.5}$$

where 1_F is a projection in h : $h_F = 1_F h$.

Let us notice that the hamiltonian $\hat{h}_F$ for the Fermi oscillator in the representation (2.4) reads

$$\hat{h}_F = \omega a^* a = \omega b^* :\exp(-b^*b): b , \tag{2.6}$$

while this for the Bose oscillator would have the form:

$$\hat{h}_B = \omega b^* b , \quad \hat{L} = \hat{L}_B = b^* \left[i \frac{d}{dt} - \omega \right] b . \tag{2.7}$$

Let furthermore $|\beta)$ be a coherent (Bose oscillator) state for the Fock representation of the CCR algebra

$$|\beta) = \exp(\beta b^* - \beta^* b)|0) . \tag{2.8}$$

As is well known, any bounded operator

$$\hat{B} = \sum_{nm} K_{nm} b^{*n} b^m \tag{2.9}$$

has its normal symbol, [19] given by

$$B = \sum_{nm} K_{nm} \beta^{*n} \beta^m = (\beta|\hat{B}|\beta) . \tag{2.10}$$

Then its functional representative (kernel) reads

$$\hat{B}[\beta^*, \beta] = B \exp \beta^* \beta . \tag{2.11}$$

The kernel of the infinitesimal operator $\hat{U}(\Delta t)$ is of main importance in the derivation, [18], of the path integral formula for Tr exp $(-iHT)$. We have

$$\hat{U}_B[\beta^*, \beta](\Delta t) = \exp(\beta^*\beta - ih_B^{cl}\Delta t), \qquad h_B^{cl} = (\beta|\hat{h}_B|\beta) = \omega\beta^*\beta\,, \tag{2.12}$$

so that the (formal continuum limit) path integral representation of Tr exp $(-i\hat{h}_B T)$ reads, [18]:

$$I_B = \mathrm{Tr}\exp(-i\hat{h}_B T) = \int [d\beta][d\beta^*] \exp i \int_0^T \{i\beta^*(t)\dot{\beta}(t) - \omega\beta^*(t)\beta(t)\}\, dt$$
$$= \int [d\beta][d\beta^*] \exp i \int_0^T L_B(t)\, dt\,, \tag{2.13}$$

with the accuracy up to the normalization factor which reflects the choice of the boundary conditions.

Knowing that the Fermi propagator $\hat{U}_F(\Delta t)$ can be represented in the Hilbert space of the Bose oscillator, we can follow step by step the just described route. Let us notice that:

$$\hat{U}_F(\Delta t) = \exp(-i\hat{h}_F\Delta t) \simeq 1_F - i\hat{h}_F\Delta t$$
$$= :\exp(-b^*b): + b^*:\exp(-b^*b):b - i\omega\Delta t b^*:\exp(-b^*b):b, \tag{2.14}$$

so that the normal symbol for $\hat{U}_F(\Delta t)$ reads

$$(\beta|\hat{U}_F(\Delta t)|\beta) \simeq \exp(-\beta^*\beta) + \beta^*\beta \exp(-\beta^*\beta) - \mathrm{i}\omega\Delta t\beta^*\beta \exp(-\beta^*\beta)\,, \tag{2.15}$$

and consequently the infinitesimal kernel is

$$\hat{U}_F[\beta^*, \beta](\Delta t) \simeq 1 + \beta^*\beta - i\omega\Delta t\beta^*\beta$$
$$= (1+\beta^*\beta)\left(1 - i\omega\Delta t\frac{\beta^*\beta}{1+\beta^*\beta}\right)$$
$$\simeq (1+\beta^*\beta)\exp\left(-i\omega\Delta t\frac{\beta^*\beta}{1+\beta^*\beta}\right)$$
$$= \exp\ln(1+\beta^*\beta)\exp\left(-i\omega\Delta t\frac{\beta^*\beta}{1+\beta^*\beta}\right), \tag{2.16}$$

where $\ln(1+\beta^*\beta)$ replaces the $\beta^*\beta$ term of (2.13) while $\omega\beta^*\beta/(1+\beta^*\beta)$ appears instead of ωb^*b.

The formal expression for Tr $\exp(-i\hat{h}_F T)$ evaluated according to the Bose oscillator recipe of [18] reads:

$$I_F = \mathrm{Tr}\exp(-i\hat{h}_F T) = \int [d\beta][d\beta^*]\exp i\int_0^T \frac{i\beta^*\dot{\beta} - \omega\beta^*\beta}{1+\beta^*\beta}\, dt$$
$$= \int [d\beta][d\beta^*]\exp i\int_0^T dt\frac{L_B(t)}{1+\beta^*(t)\beta(t)} = \int [d\beta][d\beta^*]\exp iS_F(\beta^*, \beta)\,. \tag{2.17}$$

It is a c-number alternative for the usual Grassmann algebra path integral formula, [17], which though not of a comparable calculational simplicity, does involve integrations with respect to the conventional c-number paths only.

Let us stress that the crucial difference between the Bose oscillator formula (2.13) and the Fermi one (2.17) lies in the appearance of the "damping" factor $1/(1+\beta^*\beta)$. However it implies that the only set of paths which give comparable contributions to both I_F and I_B consists of:

(a) solutions of the equation $(i(d/dt)-\omega)\beta=0$;

(b) all paths constrained to obey the restriction: $|\beta^*\beta|\ll 1$.

This is the sense in which we find it reasonable to discuss the relevance of the classical c-number problem for the construction of its quantized Fermi partner. The oscillator c-number problem does indeed manifestly contribute to the Fermi oscillator amplitudes.

We may note that the condition for I_F to be stationary is

$$\frac{\delta S_F}{\delta\beta}=0\,, \tag{2.18}$$

resulting in

$$(1+\beta^*\beta)\Big(i\frac{d}{dt}-\omega\Big)\beta=\beta^*\beta\Big(i\frac{d}{dt}-\omega\Big)\beta\,, \tag{2.19}$$

which in turn implies the conventional oscillator equations of motion.

Suppose now that we consider a countable sequence of non-interacting, ultra-locality, Fermi oscillators

$$H_k^\delta=\sum_k H_F(k)\,,\qquad H_F(k)=\omega b_k^*\,{:}\exp(-b_k^*b_k){:}\,b_k\,,$$

$$b_k=\frac{1}{\sqrt{\delta}}\int_{R^1}\chi_k(x)b(x)\,dx,\qquad [b(x),b^*(y)]_-=\delta(x-y)\,,$$

$$[b(x),b(y)]_-=0,\qquad b(x)|0)=0,\qquad \forall x\,, \tag{2.20}$$

$\chi_k(x)=1$, $x\in\Delta_k$, 0 otherwise, δ being the lattice spacing. If to take a coherent state expectation value of H_F^δ

$$(\beta|H_F^\delta|\beta)=\sum_k \omega\beta_k^*\beta_k\exp(-\beta_k^*\beta_k)\,,$$

$$|\beta)=\exp\sum_k(\beta_k b_k^*-\beta_k^*b_k)|0)\,, \tag{2.21}$$

we find that upon letting δ go to 0, the following relation appears:

$$(\beta|H_F^\delta|\beta)\simeq\sum_k\omega\delta|\beta(x_k)|^2\exp(-\delta|\beta(x)_k|^2)\,. \tag{2.22}$$

We can safely achieve the continuum limit $\delta \to 0$ thus arriving at

$$(\beta|H_F|\delta) = \omega \int dx|\beta(x)|^2 = \omega\|\beta\|^2 = (\beta|H_B|\beta) = H_{cl}, \tag{2.23}$$

where

$$H_B = \omega \int dx b^*(x)b(x),$$

$$|\beta) = \exp \int dx[\beta(x)b^*(x) - \beta^*(x)b(x)]|0), \tag{2.24}$$

which proves quite an interesting phenomenon in that while approaching the continuum we can make disappear the difference between Bose and Fermi cases: they correspond to the same classical model.

3. Chiral invariant Gross–Neveu model

Let us consider the classical c-number spinor model, which is described by the lagrangian

$$L = \bar{\psi}[i\gamma^\mu \partial_\mu - g(\sigma + i\pi\gamma_5)]\psi, \tag{3.1}$$

provided the auxiliary fields σ, π satisfy the equations of motion

$$\sigma = -g\bar{\psi}\psi, \qquad \pi = -ig\bar{\psi}\gamma_5\psi, \tag{3.2}$$

which can be summarized by considering the modified lagrangian

$$L' = L - \tfrac{1}{2}(\sigma^2 + \pi^2), \tag{3.3}$$

with the set σ, π, ψ, $\bar{\psi}$ of independent classical fields. The corresponding equations of motion describe the number N of massless Dirac fields in the external potential

$$[i\gamma^\mu \partial_\mu - g(\sigma + i\pi\gamma_5)]\psi_i = 0, \qquad i = 1, 2, \ldots, N, \tag{3.4}$$

to which the constraints (3.2) do apply.

A thorough study of the system (3.1), (3.2) is given in [15] by using the inverse scattering techniques, for the case $N = 1$ which is a free field, and the non-trivial $N = 2$ case. The soliton solutions were found.

If to admit that our spinor fields are not c-numbers but the Fermi operators

$$\psi = \psi_{\alpha a}(x), \qquad \alpha = \pm 1, \qquad a = 1, 2, \ldots, N,$$

$$[\psi_{\alpha a}(x), \psi^*_{\beta b}(y)]_+ = \delta_{\alpha\beta}\delta_{ab}\delta(x-y), \qquad [\psi_{\alpha a}(x), \psi_{\beta b}(y)]_+ = 0, \tag{3.5}$$

we arrive at the quantized chiral invariant Gross–Neveu model, whose conventional path integral representation [17, 20] would make use of the pseudoaction following from the lagrangian (3.3) but under an assumption that all spinor fields ψ, $\bar{\psi}$ take their values in the Grassmann algebra, instead of taking them in the commuting spinor function ring:

$$I_F = \int [d\sigma][d\pi] \exp\left\{-\tfrac{1}{2}i \iint_{0-\infty}^{T+\infty} (\sigma^2+\pi^2)\, dt\, dx\right\} I_F(\sigma, \pi),$$

$$I_F(\sigma,\pi) = \int [d\psi][d\bar{\psi}] \exp i \iint_{0-\infty}^{T+\infty} dt\, dx \bar{\psi}[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\psi$$

$$= \left\{ I(0) \frac{\det[\gamma^0(i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5))]}{\det(\gamma^0 i\partial\!\!\!/)} \right\}^N$$

$$= \left\{ I(0) \prod \frac{\varepsilon(\sigma + i\pi\gamma_5)}{\varepsilon(0)} \right\}^N, \tag{3.6}$$

where N is the number of color degrees and the ε's follow from the eigenvalue problem

$$\gamma^0[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\xi = \varepsilon\xi, \tag{3.7}$$

under an assumption that

$$\xi(x, t+T) = -\xi(x,t),\ \xi(x+L, t) = \xi(x,t),$$

L being finite.

Let us now notice that the lagrangian of our model can be rewritten as follows:

$$L = i\psi^*\dot{\psi} - [i\psi^*\gamma_5\partial_x\psi + g\psi^*\gamma_0(\sigma + i\pi\gamma_5)\psi] = \pi\dot{\psi} - H, \tag{3.8}$$

with

$$\pi = i\psi^*, \qquad H = \psi^*[i\gamma_5\partial_x + g\gamma_0(\sigma + i\pi\gamma_5)]\psi, \tag{3.9}$$

where ψ, ψ^* can be read out either as c-number spinors or Grassmann algebra valued spinors, if H is to be viewed (more or less) classically, and as Fermi operators in the quantum case. Then the ψ^*, ψ are supposed to generate a Fock representation of the CAR algebra, and this is the case when by using the "bosonization" recipe of [13] we are able to translate the model to the purely Bose (CCR algebra) language. The formulae arising are complicated, but they simplify if we exploit the quadratic form of the hamiltonian, noticing that when it acts on the state vectors what we must account for is not the anticommutativity property of fields at distinct space points, but the Pauli principle. This follows from the projection theorems (see especially theorem 4) of [10], where we have established the relation

$$1_F :F(\phi^*, \phi):^{c} 1_F \equiv :F(\psi^*, \psi): \tag{3.10}$$

connecting the normal ordered operator members of the Fermi field algebra $F(\bar{\psi}, \psi)$ and of the Bose field algebra, upon "bosonization" [13] of canonical fermions ψ^*,

ψ, in terms of canonical bosons ϕ^*, ϕ. In our case the underlying projection formula reads

$$H_F = 1_F H_B 1_F = 1_F \phi^* 1_F [i\gamma_5 \partial_x + g\gamma_0(\sigma + i\pi\gamma_5)] 1_F \phi 1_F \,, \tag{3.11}$$

where the commuting operators $1_F\phi^* 1_F$ appear instead of the originally used Fermi ones. Because 1_F is a projection in the Hilbert space of the Bose system, we have:

$$H_B = H_F + 1_F H_B (1-1_F) + (1-1_F) H_B 1_F + (1-1_F) H_B (1-1_F) \,, \tag{3.12}$$

which if applied to the Fermi vectors $1_F|\psi) = |\psi) \in \mathscr{H}_B$ gives

$$H_B|\psi) = H_F|\psi) + (1-1_F)H_B|\psi) \,. \tag{3.13}$$

Assume that $H_B|\psi)$ belongs to the range of the operator 1_F as also $|\psi)$; this is the case for our example, then

$$H_B|\psi) = H_F|\psi), \qquad [H_B, 1_F]_- = 0 \Rightarrow \tag{3.14}$$

$$\mathrm{Tr}\exp(-iH_B t) = \mathrm{Tr}\exp(-iH_F t) + \mathrm{Tr}\exp[-i(1-1_F)H_B(1-1_F)] \,,$$

so that the Bose trace includes the Fermi trace as a well defined, but to be extracted, contribution.

For $\mathrm{Tr}\exp(-iH_B t) = I_B(\sigma, \pi)$ one has a conventional path integration formula in terms of c-number spinor paths

$$I_B(\sigma, \pi) = \mathrm{Tr}\exp(-iH_B t) = \int [d\phi^*][d\phi] \exp iS(\phi^*, \phi, \sigma, \pi) \,,$$

$$L = \bar{\phi}[i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\phi - \tfrac{1}{2}(\sigma^2 + \pi^2) \,. \tag{3.15}$$

To compute the Fermi trace one can obviously use the Grassmann algebra methods, and the result is well known, but it is quite instructive to investigate the relationship between I_B and I_F on the c-number spinor level, like in the previous oscillator example.

Let us recall that the "stationary phase approximation" concept amounts to approximating the path integral $\int [d\phi] \exp iS(\phi)$ by the special value of the integrand, namely by $\exp i\underline{S}$ with $\underline{S} = S(\phi, \delta S/\delta\phi = 0)$. For the Bose integral we would have

$$\frac{\delta S}{\delta \phi^*} = 0 = \frac{\delta S}{\delta \phi} \Rightarrow, \qquad [i\partial\!\!\!/ - g(\sigma + i\pi\gamma_5)]\phi = 0 \,. \tag{3.16}$$

On the other hand $I_B(\sigma, \pi)$ itself should be viewed as an integrand in

$$I = \int [d\sigma][d\pi] \exp iS_{\mathrm{eff}}(\sigma, \pi) = \int [d\sigma][d\pi] I_B(\sigma, \pi) \,, \tag{3.17}$$

which by the stationary phase argument can be approximated by $\exp iS_{\mathrm{eff}}$ with $\delta S_{\mathrm{eff}}/\delta\sigma = 0 = \delta S_{\mathrm{eff}}/\delta\pi$. Since

$$\exp[iS_{\mathrm{eff}}(\sigma, \pi)] \simeq \exp\left[iS\left(\phi^*, \phi, \frac{\delta S}{\delta \phi} = 0 = \frac{\delta S}{\delta \phi^*} = \frac{\delta S}{\delta \phi^*}\right)\right]$$

we arrive at

$$I \simeq \exp i\underline{S}_{\text{eff}} = \exp\left[iS\left(\phi^*, \phi, \sigma, \pi; \frac{\delta S}{\delta\phi} = 0 = \frac{\delta S}{\delta\phi^*} = \frac{\delta S}{\delta\sigma} = \frac{\delta S}{\delta\pi}\right)\right], \quad (3.18)$$

but this implies that the underlying ϕ^*, ϕ do satisfy the original classical (c-number) field equations of the chiral invariant Gross–Neveu model, i.e. as expected the approximation is realized by the classical fields, as it should be in the Bose theory.

Let us fix a specific solution χ^*, χ, $\sigma = -g\bar{\chi}\chi$, $\pi = -ig\bar{\chi}\gamma_5\chi$, then by computing

$$I_{\text{B}}(\sigma, \pi)[\chi^*, \chi] = \int [\text{d}\phi][\text{d}\phi^*] \exp\left(iS[\phi^*, \phi, \sigma(\chi^*, \chi), \pi(\chi^*, \chi)]\right), \quad (3.19)$$

one gives account of quantum fluctuations about the stationary phase solution, with potentials being kept in their zero-loop order.

For the Fermi contribution I_{F} to I_{B} the stationary phase argument is not applicable immediately. Let us however make use of (3.11). First let us notice that the continuum theory level can be replaced by the appropriate (usual discretization schemes do not seem to apply) lattice one

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (3.20)$$

$$\begin{aligned} H_{\text{F}}(x) &= 1_{\text{F}}\phi^*[i\gamma_5\partial_x - g\gamma_0(\sigma + i\pi\gamma_5)]\phi 1_{\text{F}} \\ &= 1_{\text{F}}[i(-\phi_1^*\partial\phi_2 + \phi_2^*\partial\phi_1) \\ &\quad -g\sigma(\phi_1^*\phi_1 - \phi_2^*\phi_2) - g\pi(\phi_1^*\phi_2 - \phi_2^*\phi_1)]1_{\text{F}}, \end{aligned}$$

$$\begin{aligned} H_{\text{F}}(x) \to\ & 1_{\text{F}}^k[i(-\phi_1^*(k)\partial_\alpha\phi_2(k,\alpha)_{|\alpha=0} + \phi_2^*(k)\partial_\alpha\phi_1(k,\alpha)_{|\alpha=0}) \\ & -g\sigma(\phi_1^*(k)\phi_1(k) - \phi_2^*(k)\phi_2(k)) - g\pi(\phi_1^*(k)\phi_2(k) - \phi_2^*(k)\phi_1(k)) \Rightarrow \\ & H_{\text{F}}(x) \to H_{\text{F}}(k) = 1_{\text{F}}^k H_{\text{B}}(k) 1_{\text{F}}^k, \end{aligned}$$

where

$$\phi_i^*(k) = \frac{1}{\sqrt{\delta}} \int \text{d}x\chi_k(x)\phi_i^*(x) \simeq \sqrt{\delta}\,\phi_i^*(x), \quad x \in \Delta_k,$$

$$\phi_i^*(k, \alpha) = \frac{1}{\sqrt{\delta}} \int \text{d}x\chi_k(x)\phi_i^*(x + \alpha),$$

$$1_{\text{F}}^k = \prod_{i=1}^{2} \{:\exp(-\phi_i^*(k)\phi_i(k)): + \phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)): \phi_i(k)\},$$

$$1_{\text{F}} = \prod_{k} 1_{\text{F}}^k, \quad (3.21)$$

with

$$
\begin{aligned}
1_F\phi_i^*(k)1_F &= 1_F^k\phi_i^*(k)1_F^k = \sigma_i^*(k) = \phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)): ,\\
1_F\phi_i(k)1_F &= 1_F^k\phi_i(k)1_F^k = \sigma_i(k) = :\exp(-\phi_i^*(k)\phi_i(k)): \phi_i(k) ,\\
1_F\partial_\alpha\phi_i(k)_{|\alpha=0}1_F &= \partial_\alpha\, 1_F^k\phi_i(k,\alpha)1^k_{F|\alpha=}0: = \partial_\alpha\sigma_i(k,\alpha)_{|\alpha=0} ,\\
\sigma_i(k,\alpha) &= :\exp(-\phi_i^*(k,\alpha)\phi_i(k,\alpha)): \phi_i(k,\alpha) .
\end{aligned}
\tag{3.22}
$$

The above definitions allow for the following computation:

$$
\begin{aligned}
&\partial_\alpha\sigma_i^*(k)\sigma_i(k,\alpha)_{|\alpha=0}:\\
&\quad = \phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)):\\
&\qquad \times\partial_\alpha\{:\exp(-\phi_i^*(k,\alpha)\phi_i(k,\alpha)): \phi_i(k,\alpha)\}_{|\alpha=0}\\
&\quad = \phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)): \partial_\alpha\phi_i(k,\alpha)_{|\alpha=0}\\
&\qquad -\phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)): \{[\partial_\alpha\phi_i^*(k,\alpha)_{|\alpha=0}]: \exp(-\phi_i^*(k)\phi_i(k): \phi_i^2(k)\\
&\qquad +\phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)): (\partial_\alpha\phi_i(k,\alpha))_{|\alpha=0}\}\cdot\phi_i(k)\\
&\quad = \phi_i^*(k) :\exp(-\phi_i^*(k)\phi_i(k)): \partial_\alpha\phi_i(k,\alpha)_{|\alpha=0} ,
\end{aligned}
\tag{3.23}
$$

and for this it was necessary to have defined the kinetic term of the hamiltonian. Consequently:

$$
\begin{aligned}
H_F &= \int dx H_F(x) \to H_F^\delta = \sum_k H_F(k),\\
H_F(k) &= -i\phi_1^*(k) :\exp\Big(-\sum_j \phi_j^*(k)\phi_j(k)\Big): \partial_\alpha\phi_2(k,\alpha)_{|\alpha=0} + i\phi_2^*(k)\\
&\quad :\exp\Big(-\sum_j \phi_j^*(k)\phi_j(k)\Big): \partial_\alpha\phi_1(k,\alpha)_{|\alpha=0}\\
&\quad -g\sigma[\phi_1^*(k): \exp(-\phi_1^*(k)\phi_1(k)):\phi_1(k)\\
&\quad -\phi_2^*(k): \exp(-\phi_2^*(k)\phi_2(k)): \phi_2(k)]\\
&\quad -g\pi\Big[\phi_1^*(k): \exp\Big(-\sum_j \phi_j^*(k)\phi_j(k)\Big): \phi_2(k)\\
&\quad -\phi_2^*(k):\exp\Big(-\sum_j \phi_j^*(k)\phi_j(k)\Big): \phi_1(k)\Big] .
\end{aligned}
\tag{3.24}
$$

The infinitesimal propagator for the discretized hamiltonian reads

$$
\begin{aligned}
\hat U_F^\delta(\Delta t) &= \exp(-iH_F^\delta\Delta t) \simeq \prod_k (1_F^k - iH_F^k\Delta t)\\
&= \prod_k \hat U_F^\delta(k,\Delta t) ,
\end{aligned}
\tag{3.25}
$$

so that its infinitesimal kernel is

$$U_F^\delta(\Delta t) = \left[\exp \sum_k \sum_{i=1}^{2} \beta_i^*(k)\beta_i(k)\right]\left(\beta\left|\prod_k (1_F^k - iH_F^k \Delta t)\right|\beta\right)$$

$$= \prod_k \left\{\prod_{i=1}^{2} [1 + \beta_i^*(k)\beta_i(k)] - i\Delta t H_{cl}^k\right\},$$

$$H_{cl}^k = -i\beta_1^*(k)\partial_\alpha\beta_2(k,\alpha)_{|\alpha=0} + i\beta_2^*(k)\partial_\alpha\beta_1(k,\alpha)_{|\alpha=0}$$
$$-g\sigma\{\beta_1^*(k)\beta_1(k)\exp[\beta_2^*(k)\beta_2(k)] - \beta_2^*(k)\beta_2(k)\exp[\beta_1^*(k)\beta_1(k)]\}$$
$$-g\pi[\beta_1^*(k)\beta_2(k) - \beta_2^*(k)\beta_1(k)]. \tag{3.26}$$

If now we exploit our assumption $\delta \ll 1$ (which anticipates letting δ go to 0) we arrive at

$$1 + \beta_i^*(k)\beta_i(k) \to 1 + \delta\beta_i^*(x)\beta_i(x), \qquad x \in \Delta_k,$$
$$\exp(\beta_i^*(k)\beta_i(k)) \to \exp \delta\beta_i^*(x)\beta_i(x).$$

Then we have

$$U_F^\delta(\Delta t) \simeq \prod_k \prod_{i=1}^{2} [1 + \delta\beta_i^*(x_k)\beta_i(x_k)]\left\{1 - \frac{i\Delta t\delta H_{cl}(x_k)}{\prod_i [1 + \delta\beta_i^*(x_k)\beta_i(x_k)]}\right\}$$

$$\simeq \prod_k \prod_i [1 + \delta\beta_i^*(x_k)\beta_i(x_k)][1 - i\Delta t\delta H_{cl}(x_k)]$$

$$\simeq \exp \sum_k \left\{\delta \sum_i \beta_i^*(x_k)\beta_i(x_k) - i\Delta t\delta H_{cl}(x_k)\right\}$$

$$\to \exp \int dx \left\{\sum_i \beta_i^*(x)\beta_i(x) - i\Delta t H_{cl}(x)\right\}, \tag{3.27}$$

provided we consider these β_i^*, β_i's only, which are discretizations of sufficiently regular spinor trajectories

$$\beta_i^*(x)\beta_i(x) \le A < \infty, \qquad A \in R^+, \qquad \forall i = 1, 2, \qquad x \in \mathrm{R}^1, \tag{3.28}$$

since then only can we safely neglect contributions from $\delta\beta_i^*(x)\beta_i(x)$ as we made in the above derivation.

Consequently as far as path integrations necessary to compute $\mathrm{Tr}\exp(-iH_F t)$ are performed with respect to a subset of trajectories $\beta_i^*(x)\beta_i(x) \le A < \infty$, the corresponding contribution to I_F coincides with this to I_B. The stationary phase approximation then reveals the same classical spinor solutions of the chiral invariant Gross–Neveu model field equations, irrespective of whether we consider I_B or I_F.

Let us emphasize that such a restricted continual integral cannot recover the whole of I_F, since just the irregular trajectories forbidding the neglect of contributions from $\delta\beta_i^*(x)\beta_i(x)$ quantally make a difference between the Bose and Fermi cases. But then we do not have a reasonable continuum limit and the only correct path integral representation for I_F in terms of c-number paths is the lattice one. One should realize that in any rigorous path integral computation the discretization is in fact unavoidable.

The above observations have no relevance as far as the explicit standard computations are concerned on the Fermi field level, but seem to be of the foundational nature, see e.g. [14, 21].

Our conclusion is that the classical c-number chiral invariant Gross–Neveu model is quite a reasonable classical partner for the Fermi quantized chiral invariant Gross–Neveu model, and there is a non-trivial relationship between both, contrary to the belief that no deeper relationship exists.

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