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Introduction to the Lattice Quantization of the Sine-Gordon System

by

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Summary. We perform a quantization of the Sine–Gordon 1-soliton in so called single-site approximation, when continuous pulses are approximately reproduced in the linear lattice of classical plane pendula. We prove the existence of the causal time evolution operator for quantum 1-solitons. The energy operator is not capable to generate the solitary motion.

1. In the traditional derivation of the quantum field theory of a free scalar field, the Hamiltonian (two space-time dimensions are taken for simplicity):

$$H=\int dx \left\{\frac{1}{2}\left(\frac{\partial \Phi}{\partial t}\right)^2+\frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2+\frac{1}{2}\,\mu^2\,\Phi^2\right\}(x,t), \tag{1.1}$$

can be approximated on the finite linear lattice, by the Hamiltonian:

$$H=\Lambda\sum_s\left\{\frac{1}{2}p_s^2+\frac{1}{2}(\nabla x_s)^2+\frac{1}{2}\,\mu^2\,x_s^2\right\}, \tag{1.2}$$

with s enumerating the lattice sites.

The omission of the gradient part reduces the problem to its single site approximation by the linear chain of harmonic oscillators. In this approximation the quantization of the system lies in introducing quantum oscillators in the place of the classical ones. To restore the complete quantum system we must here perform a translation to the quantum language of the neighbour interaction (gradient) term. One makes it according to [1]:

$$H=\Lambda\sum_s\left\{\frac{p_s^2}{2}+\frac{1}{2}\left(\frac{\mu^2}{\Lambda^2}+D(0)\,x_s^2\right)\right\}+\frac{1}{2}\sum_{s_1\neq s_2}D(s_1\,s_2)\,x_{s_1}\,x_{s_2}. \tag{1.3}$$

Quite analogous procedure can be repeated in the case of the Sine–Gordon system. The corresponding Hamiltonian:

$$H=\int\frac{1}{2}\left\{\left(\frac{\partial \Phi}{\partial t}\right)^2+\left(\frac{\partial \Phi}{\partial x}\right)^2+\lambda(1-\cos\Phi)\right\}(x,t), \tag{1.4}$$

is approximated on the linear lattice by:

$$H = A \sum_s \left\{ \frac{1}{2} [\pi_s^2 + (\nabla \Phi_s)^2] + \lambda (1 - \cos \Phi_s) \right\}, \tag{1.5}$$

where again the gradient term is in fact the interaction part of the Hamiltonian, and carries the nearest neighbour coupling. Its omission leaves us with the linear chain of the independent plane pendula, which was the root for the construction of the Scott's mechanical analog transmission line [2, 3] for the Sine–Gordon pulses. We do not pretend here to get exact solutions of the discrete problem. We rather wish to find a quantum image of 1-soliton solutions in the single-site approximation together with the quantum term of the Hamiltonian which gives account of the classically observed long-range correlations between nearest neighbours in the chain.

2. The knowledge of the classical potential $V = mgl(1 - \cos\theta)$ implementing a movement of the plane pendulum with the length l and mass m, allows to consider the first quantization of this problem in the form of the Schroedinger equation [4, 5]:

$$\left[-\frac{\hbar^2}{2ml^2} \frac{d^2}{d\theta^2} + mgl(1 - \cos\theta) \right] \psi = E\psi, \tag{2.1}$$

where we restricted considerations to the stationary case only. Putting $2z = \theta$ we trivially receive:

$$\left[\frac{d^2}{dz^2} + (a - 2q\cos 2z) \right] \psi = 0, \tag{2.2}$$

where

$$q = 4m^2 l^3 g/\hbar^2 \qquad a = \frac{8ml^2}{\hbar^2}(E - mgl). \tag{2.3}$$

An equation (2.2) is known as Mathieu equation and is a solvable problem. Mathieu functions can be proved to constitute a complete orthonormal set in $\alpha^2\,(0, 4\pi)$ which is thus a Hilbert space of pendulary states. Creation and annihilation operators can be here constructed in the form of tensor products. In this case let us introduce quantities:

$$a^* = \sum_{k=0}^{\infty} \sqrt{k+1}\, e_{k+1} \otimes e_k, \qquad a = \sum_{k=1}^{\infty} \sqrt{k}\, e_{k-1} \otimes e_k \tag{2.4}$$

the action of which makes invariant the appropriate domain $D \subset h$

$$a^* e_n = \sum_{k=0}^{\infty} \sqrt{k+1}\, e_{k+1} (e_{k+1}, e_n) = \sqrt{n+1}\, e_{n-1}, \qquad a e_n = \sum_{k=1}^{\infty} \sqrt{k}\, e_{k-1} (e_k, e_n) = \sqrt{n}\, e_{n-1}. \tag{2.5}$$

Here e_k, e_j are the elements of the real orthonormal complete system in $\alpha^2(0, 4\pi)$. We have

$$[a, a^*]=\sum_{i=0}^{\infty} e_i\otimes e_i=\mathbf{1}, \tag{2.6}$$

$$[a, a]=0=[a^*, a^*], \tag{2.7}$$

$$ae_0=0. \tag{2.8}$$

We have thus proved that the triple $\{a, a^*, e_0\}$ spans in $D\subset h$ a Fock representation of the *CCR* algebra where e_0 is a vacuum vector for this representations.

The quantum mechanical Hamiltonian though rather not admitting any reasonable number of particles representation, can always be considered in the matrix form:

$$H=\sum_j E_j\, e_j\otimes e_j. \tag{2.9}$$

Remark. Let us mention that classically plane pendulum problem can always be, in a suitable limit, restricted to the harmonic oscillator problem. On the qunatum level this is not so, because in fact the energy eigenvalues can tend to oscillatory eigenvalues, but the double degeneracy appears here. Each oscillator level corresponds to the two adjacent quantum pendulum eigenvalues. In the case of the eigenfunctions we deal with the Klauder's phenomenon where the limitting ($q\to\infty$) eigenfunctions correspond to the pseudo-free quantum oscillator [6].

An extension of the above results onto quantum field theory is immediate. One must here follow the conventional construction [7, 8] which allows to introduce the so called product representation of the CCR algebra for which the carrier space is a corresponding IDPS (incomplete direct product space), Fock space, in our case.

3. Following the preliminary formulation [9] we shall now perform the lattice quantization of the Sine–Gordon 1-solitons. They are the solutions of the equation:

$$\Box\Phi(x, t)=\left(\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial t^2}\right)\Phi(x, t)=m^2\sin\Phi(x, t), \tag{3.1}$$

which are of the form:

$$\Phi(x, t)=4\tan^{-1}\exp\left(\pm m\frac{x-vt}{\sqrt{1-v^2}}\right). \tag{3.2}$$

The energy $E=\dfrac{8m}{\sqrt{1-v^2}}$ and momentum of the soliton can be easily calculated:

$$p=\frac{8mv}{\sqrt{1-v^2}} \qquad c=\hbar=1.$$

The approximation of the 1-soliton pulse on the linear lattice is given immediately, if with each site (the spacing is a), to associate a corresponding characteristic function

$$\Delta_s \leftarrow \chi_s(x) = \begin{cases} 1 & x \in \Delta_s \\ 0 & x \notin \Delta_s \end{cases}, \tag{3.3}$$

$$\Delta_s \cap \Delta_t = \emptyset \quad \text{for } s \neq t, \quad \bigcup_{s=0}^{\infty} \Delta_s \subseteq R^1, \quad \mu(\Delta_s) = a \ll 1$$

so that

$$\Phi_s(t) := \frac{1}{a} \int_{R'} dx \chi_s(x)\, \Phi(x, t),$$

$$\mathscr{L}_s(t) := \frac{1}{a} \int_{R'} dx \left\{ \frac{1}{2} [\Phi_x^2 - \Phi_t^2 + 2m^2 (1 - \cos \Phi)] \right\} (x, t)\, \chi_s(x), \tag{3.4}$$

$$\mathscr{H}_s(t) := \frac{1}{a} \int_{R'} dx \left\{ \frac{1}{2} [\Phi_x^2 + \Phi_t^2 + 2m^2 (1 - \cos \Phi)] \right\} (x, t)\, \chi_s(x).$$

Having given the energy density of the 1-soliton $\mathscr{H}(x, t) = \mathscr{H}(\Phi)(x, t)$ one can establish a position of the energy cetre of the pulse at the initial instant of time $t = 0$. Let us assume that this particular point, the collective variable, belongs to the 0-th site which is identified with the $s = 0$ interval. In consequence the 1-soliton is completely described by the following collection of the initial data

$$\Phi_s(0) = \varphi(y + sa) = \varphi_s(y), \qquad \pi_s(y) = \dot{\Phi}_s(0) = \pi(y + sa),$$

$$E = \int_{R'} dx \mathscr{H}(\Phi)(x, 0) \cong \sum_s \mathscr{H}(\Phi)_s(y). \tag{3.5}$$

Here $\varphi_0(y) = \varphi(y)$, $\pi_0(y) = \pi(y)$ and all the data tend to $\varphi(y)$ and $\pi(y)$ respectively with $|y| \to \infty$

From now on, we shall simplify considerations by omitting the collective variable y in all the formulas: $\varphi_s(y) = \varphi_s$; $\pi_s(y) = \pi_s$, $E \cong \sum_s \mathscr{H}_s$. The uniform motion rule $\Phi(x - vt) = \Phi(x, t)$ which holds on the continuous level is now approximated by the following motion rule of the set of the initial data:

$$\varphi_s(t) = \varphi_s(y, t) = \varphi y - vt + sa) \Rightarrow \varphi_s(t) = \varphi_{s-n} \quad (t = \tfrac{na}{v})$$

$$\pi_s(t) = \pi_s(y, t) = \pi(y - vt + sa) \Rightarrow \pi_s(t) = \pi_{s-n} \tag{3.6}$$

which is simply the shift of the data along the chain, following from the influence of the neighbour coupling, implied by the gradient term. We have thus separated on the classical level the nonlinear geometry (shape) of the solution from the fully linear dynamics. Let us add that a similar procedure can be repeated in the case of N-soliton solutions where the number N of collective variables is necessary [13, 14].

Let now the quantum chain be given, where in the single site approximation, a sequence of independent quantum pendula appears, together with a corresponding single-site basis. We shall translate the classical data and the motion rule to the quantum lattice.

Let us begin from the question of statistics. Because each site of the lattice is occupied by a single quantum pendulum whose spectrum is positive and nondegenerate, if we pretend to describe the line of quantum pendula, the Pauli exclusion principle should govern its behaviour: the occupation number of each (s, n)-th state of the lattice is either 1 or 0: (s, n) means that n-th energy level of the quantum pendulum is occupied at the s-th site.

Because there is no immediate need to have a reasonable correspondence between the classical and Fermion level, we shall formulate all the results for the subsidiary mediating Boson level and then in the sense of the weak excitation limit, the transition to the final Fermion variables will be performed, see e.g. [12]. Let us denote by E_0, E_1 the energies of the two lowest stationary levels of the quantum pendulum. We have mapped each plane pendulum, whose energy $\mathscr{H}_s$ does not exceed $E_{\min}=E_1-E_0$ into a nonexcited, hence occupying the ground state E_0, quantum pendulum. This receipt is motivated by the naive hope that such energies cannot be quantized, and play in the theory the role of an inessential noise. Now we have the question of an energy sharing between quantum pendula of the net energy E of 1-solito, which we consider as the net in the sense of the renormalization by the substraction of the ground state energy from the total energy at each site of the lattice. We expect E to be approximated by the sum of quantized portions $E_{(s,n)}$

$$\sup_{\{(s,n)\}} \sum_{(s,n)} \{E_{(s,n)}-E_0\} \cong E. \tag{3.7}$$

Because E is a macroscopic value, the equality in fact holds. The 1-soliton pulse has a finite energy value, what if combined with the requirement (3.7) clearly requires at most finite number of quantum pendula to be simultaneously excited. Note that the free field techniques, especially the Fock space methods, can be used by virtue of this argument.

With each single lattice site, let us now associate the subsidiary Boson field Φ_s, whose lattice Fourier expansion is

$$\Phi_s = \frac{1}{\sqrt{v}} \sum_k \{a_k^x \exp\left(\frac{i\kappa\pi s}{V}\right) + a_k \exp\left(\frac{-ik\pi s}{V}\right)\}, \tag{3.8}$$

which allows to introduce the corresponding creation and annihilation operators:

$$\begin{aligned} &[a_k, a_l^*]_- = \delta_{kl} \\ &[a_k, a_l]_- = 0 \qquad a_k \Omega_B = 0 \qquad \text{for all } k. \end{aligned} \tag{3.9}$$

In the above the normalization constant is $V=\dim\{(s, n)\}$ for the set of pairs realizing supremum in (3.7) and k enumerates the finite set of degrees of freedom (energy levels of pendula reproducing the 1-soliton pulse). The quantum numbers k

are defined by the initial 1-soliton data, if we define the appropriate correspondence rule, by the use of the coherent state methods:

$$|f\rangle=\exp\left\{\sum_{\kappa} f_k^+ a_k^+\right\}\Omega_B\cdot\exp\left(\frac{1}{2}\|f\|^2\right),$$

$$f_s=\frac{1}{\sqrt{v}}\sum_k\left\{f_k^+\exp\left(\frac{ik\pi s}{V}\right)+f_k^-\exp\left(-\frac{ik\pi s}{V}\right)\right\}. \tag{3.10}$$

Let us notice that putting $f_s=\varphi_s$ we get

$$\langle\varphi|\Phi_s|\varphi\rangle=\varphi_s. \tag{3.11}$$

We can expect the existence of the proper $\dot{\Phi}_s$ such that $\langle\varphi\,|\dot{\Phi}_s|\,\varphi\rangle=\pi_s$ however for this purpose we must realize the solitary dynamics in the quantum chain.

In the single-site approximation the form of the energy operator immediately follows:

$$H_s=\sum_{s'} a_s^*\left\{\frac{1}{V}\sum_k \varepsilon_k\exp\left[i\frac{k\pi}{V}(s-s')\right]\right\}a_{s'},$$

$$H=\sum_s H_s, \tag{3.12}$$

where ε_k must still be properly defined. A total energy operator for the 1-soliton reads then:

$$H=\sum_k a_k^* a_k\,\varepsilon_k. \tag{3.13}$$

The solitary evolution rule on the classical level implies:

$$\varphi_s(t)=\varphi(y+sa-vt)\;\Rightarrow\;\varphi_s(t)=\varphi_{s-n},\quad t=\frac{na}{v}$$

$$\varphi_{s-n}=\frac{1}{\sqrt{V}}\sum_k\left\{\varphi_k^+\exp^{\frac{ik\pi}{V}(s-n)}+\varphi_k^-\exp^{\frac{ik\pi}{V}(n-s)}\right\}, \tag{3.14}$$

so that

$$\varphi_k^+\left(t=\frac{na}{v}\right)=\varphi_k^+\exp\left(-\frac{ikv}{V}n\right). \tag{3.15}$$

In consequence, for $t=\dfrac{na}{v}$

$$H(t)=\sum_k a_k^*(t)\,a_k(t)\,\varepsilon_k \tag{3.16}$$

we have

$$(\varphi|H(t)|\varphi)=(\varphi|H|\varphi)\cong E. \tag{3.17}$$

On the other hand,

$$H=\sum_k H_k$$

$$(\varphi\,|H_k|\,\varphi)=\left(\varphi\,|H_k\left(t=\frac{na}{v}\right)|\,\varphi\right).$$

If now we simultaneously require:

$$\mathscr{H}_s=\frac{2}{\sqrt{V}}\sum_k \mathscr{H}_k\cos\frac{k\pi s}{V},$$
$$H_s=\frac{2}{\sqrt{V}}\sum_k H_k\cos\frac{k\pi s}{V}, \tag{3.19}$$

then the correspondence rule $\mathscr{H}_s=(\varphi\,|H|\,\varphi)$ establishes the following connection between the classical and quantum energy data:

$$\varepsilon_k=\frac{\mathscr{H}_k}{\varphi_k^+\,\varphi_k^-}. \tag{3.20}$$

To get a quantum image of the 1-soliton evolution it is useful to know that if the quantum gradient term is taken in the form [10]

$$\mathscr{D}^2_{nm}=\mathscr{D}^2(n-m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}k^2\exp\left[ik(m-n)\right],$$
$$a\to 0\Rightarrow\mathscr{D}^2\to-\nabla^2\,\delta(x-y), \tag{3.21}$$

then an immediate quantum lattice analogue of the space translation operator can be given:

$$\pi_n=-\frac{i}{\sqrt{V}}\sum_k\left\{a_k\,e^{\left(\frac{ik\pi}{V}n\right)}-a_k^*\,e^{\left(-\frac{ik\pi}{V}n\right)}\right\}k_0,$$
$$P=-i\sum_{nm}\pi_n\,\mathscr{D}_{nm}\,\Phi_m\qquad[\pi_n,\,\Phi_m]_-=-i\delta_{nm} \tag{3.22}$$

so that:

$$\exp(iP_n)\cdot\Phi_m\exp(-iP_n)=\Phi_{m-n},$$
$$\exp(iPvt)\,\Phi_m\exp(-iPvt)=\Phi_m(t)=\Phi_{m-vt}\underset{t=\frac{na}{v}}{\Rightarrow}\Phi_{m-n}. \tag{3.23}$$

Obviously, in the sense of the correspondence principle (3.23) is the quantum image of the 1-soliton evolution rule, which seems at first sight to contradict the ordinary expectations that an energy operator H should play this role rather: $(\varphi\,|\Phi_{m-n}|\,\varphi)=$ $=\varphi_{m-n}$. Let us in this place prove that the energy operator of the just constructed "quantum soliton" cannot be a correct generator of the solitary time translations.

For this purpose it suffices to notice that H obeys the restrictions of the Borchers theorem [11]. Given a one parameter group $U_t=\exp(-iHt)$ where the generator $H\geqslant c>-\infty$. Denote $F_t=U_t FU_t^{-1}$ for any operator F. If there is a pair of projectors E, F such that for $|t|<\varepsilon$, $EF_t=0$, then for any $t\in R^1$, $EF_t=0$. Let us remark that the solitary evolution rule:

$$t\to t-\frac{na}{v}\Rightarrow\varphi_s(t)\to\varphi_{s-n}(t), \tag{3.24}$$

can be equivalently described by the motion of the localization volumes (sites) while the 1-soliton not evolving at all:

$$\Delta_s\to\Delta_{s-n}\Rightarrow\varphi_s(t)\to\varphi_{s-n}(t). \tag{3.25}$$

In the operator language it has to be associated with each Δ_s, a corresponding projector E_{Δ_s}.

Let us consider the three sites Δ_s, Δ_{s-1}, Δ_{s-2}. Then obviously $E_{\Delta_s}E_{\Delta_{s-2}}=0$ and one needs at least finite time interval $|t|\geqslant\varepsilon$ to get $E_{\Delta_{s-2}}U_tE_{\Delta_s}U_t^{-1}\neq 0$ where for $|t|<\varepsilon$ $0=E_{\Delta_{s-2}}U_tE_{\Delta_s}U_t^{-1}$ holds. In consequence, for neither time t we can get the required transition

$$\Delta_s\to\Delta_s^t=\Delta_{s-2} \tag{3.26}$$

if the positive evolution operator is used. In this connection let us notice that the correct evolution operator $-Pv$ for the quantum image of our 1-soliton is manifestly not positive. The above arguments justify, in a sophisticated way, the independence of the single sites of the lattice for all times, like this appearing if the gradient term is absent in the Hamiltonian. Does it at all exclude the long range correlations for any class of positive Hamiltonians?

The above considerations suggest that together with the collective shift operator P, one should introduce a collective velocity operator $\dot{Q}$, which in the case of 1-solitons is not proportional to P as in that case $P\dot{Q}$ would be λP^2 and hence positive if $\lambda>0$. The last step in our considerations is now to make a transition to Fermion variables, which should appear by virtue of the built in Pauli exclusion principle. The most convenient here, though obviously not unique tool, seems to employ the weak excitation limit concept, and then the map [12]:

$$\begin{aligned} &1_F\Phi_s 1_F=\overline{\psi}_s,\\ &1_F H1_F=H_F,\\ &1_F P1_F=P_F, \end{aligned} \tag{3.27}$$

which realizes the translation of our construction of the quantum 1-soliton to the Fermion language.

In this place let us mention that the presented quantization procedure is the first step for the lattice quantization of the Sine–Gordon system.

The next step is to modify the lattice approximation idea to be applicable to bions and the general N-soliton solutions of the Sine–Gordon equation. We shall here

only announce that in the bion case in addition to the lattice causal evolution (hoping motion which translates excitations from one lattice site to another we are forced to introduce the acausal evolution generated by the energy operator. The last motion follows from internal oscillations, which are characteristic for bion solutions. The general case of N-soliton solutions will be considered in the forthcoming paper [13].

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П. Гарбачевски, З. Попович, **Введение в решеточную квантизацию системы Sine-Gordon**

Содержание. В работе описан метод квантования односолитонных решений уварнений–в так называемой одно-узловой аппроксимации. Найден причинный оператор временной эволюции для квантовых односолитонов. Доказанос что это негапильтоновская эволюция.