Rotational diffusions as seen by relativistic observers

Piotr Garbaczewski
Department of Physics, Brown University, Providence, Rhode Island 02912

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The major unsolved problem in the framework of Nelson’s stochastic mechanics is addressed and an attempt is made to provide a description of relativistic spin-1/2 particles in terms of Markovian diffusions on $S_3$. Random rotations are here labeled by the proper time of a particle in relativistic motion and are continuously distributed along a space-time trajectory followed by the particle in Minkowski space.

I. INTRODUCTION: THE PROBLEM OF RANDOM ROTATIONS IN SPECIAL RELATIVITY

The description of spin-1/2 in the framework of Nelson’s stochastic mechanics involves a harmonic analysis on the group $G$ of rotations in $R^3$. For functions on the group the Hilbert space structure is induced by the invariant Haar measure on $S_3$. The scalar product reads as

$$ \langle f_1, f_2 \rangle = \int d^3g f_1(g) \overline{f_2}(g) $$

where $f_1$ and $f_2$ are harmonic functions on $S_3$, and $\overline{f_2}$ is their complex conjugate. The unitary representation of the group $S_3$ is characterized by the familiar SU(2) harmonics.

Let $G(t)$ be a random variable taking values in $S_3$, which undergoes a nondissipative (rotational) Markovian diffusion. The $i$th SU(2) harmonic describes the $i$th state of stationary diffusion, $i=1,2,3,4$. For each state we can introduce a vector-valued function of this random variable $L_i = L_i(G(t))$, which following Refs. 1 and 2 may be attributed the role of an angular momentum induced by the rotation $G(t)$ in a given state of rotational diffusion. Let $e_i(g)$ denote the SU(2) harmonic for the spin-1/2 case. Then $|e_i(g)|^2$ stands for the probability distribution of $G(t)$ in the state $e_i(g)$. We can evaluate the expectation values:

$$ \langle L_i \rangle = \int L(g) |e_i(g)|^2 dg = \frac{\hbar}{2} s $$

where $s$ is the unit (spin polarization) vector in $R^3$, identifying the direction of the quantization axis in space. In the standard quantum mechanical lore it is the direction on which spin projections are equal to $+\hbar/2$.

In Dankel’s paper, the case of $s=k$ (the z direction in the Cartesian frame) was investigated. In virtue of Ref. 2 the mere change of Euler parametrization allows us to consider quite arbitrary $s$, eventually allowing for a smooth time dependence $s=s(t)$ characteristic for the spin precession.

According to Ref. 2 rotational diffusions characterizing the spin-1/2 particle at rest involve the Cartesian system frame, whose orientation relative to the laboratory frame is given by the Euler angles $(\theta, \phi, \psi) = \bar{g}$. They determine the quantization axis direction,

$$ R(\bar{g}) = \mathbf{s} = (\sin \psi \sin \theta \cos \psi \sin \theta \cos \theta) $$

All random fluctuations (e.g., rotations) are intrinsic to the system frame, and described in terms of another set of Euler angles $(\theta, \phi, \psi) = g$ referring to intrinsic rotation axes $e_\phi, e_\theta, e_\psi$ in the system frame.

The discussion of random rotations implementing spin-1/2 is purely nonrelativistic and effectively confined to the system rest frame (inhomogeneous magnetic fields alter this picture).

We denote by $K'$ the inertial rest frame, in which the spin-1/2 rotational diffusion takes place. We admit, furthermore, that $K'$ moves uniformly with the velocity $v$, $|v| < c$ relative to another inertial frame, and address the following relativistic problem.
How does the K observer perceive the Markovian diffusion taking place in K'? Is it a stochastic diffusion process again?

The nonrelativistic treatment of Ref. 2 suggests that we should first establish the transformation properties of the polarization axes when passing from one inertial frame to another. The issue has been solved in the context of the Bargmann–Michel–Telegdi equation. Specification of the rest frame polarization is known to determine the components of the polarization four-vector in any inertial frame. Indeed, if s is the rest frame polarization, its K frame image (via the Lorentz transformation \( \Lambda^{-1} \) taking \( (c,0,0) \) into \( (\gamma c,\gamma v) \) with \( \gamma = \sqrt{1 - \beta^2} \), \( \beta = v/c \)), is given by

\[
\begin{align*}
\cos \theta &= s \hat{s} = s/|s|, \\
\hat{s} &= 1 + \frac{\gamma^2}{1 + \gamma} (\beta s)^2, \\
|s| &= \left| 1 + (\beta s)^2 \right|^{1/2} \left[ \beta^2 \gamma^2 + 1 \right]^{1/2},
\end{align*}
\]

(1.7)

and \( \theta \) is uniquely defined, given \( s \) and \( v \).

Once in \( \mathbb{R}^3 \), passing from \( s \) to \( S \) amounts to a rotation by an angle \( \theta \) about the axis \( s \times \beta \) (the same as about \( S \times \beta \)),

\[
g_s s = S = g_s \beta s.
\]

(1.8)

From now on, the directions \( s \) and \( S \) will be the fixed z-axis directions in the system frames located in \( K' \) and \( K \), respectively.

Consider a unit vector \( n \), initially along the z axis of the system frame in \( K' \). Let us execute a rotation \( n \xrightarrow{\text{g}} gn \).

In view of (2.4), we have here

\[
ng_n = n + \frac{\gamma^2}{1 + \gamma} (\beta n)\beta' = g_{n_\theta},
\]

(1.9)

where \( \beta' = g_{-1} \beta \) induces a rotation of angle \( \nu' \) about the axis \( n \times \beta' \). Accordingly, a unit vector \( \hat{N}_n \) is recovered.

\[
\hat{N}_n = g_{n_\theta} = (g_{1_\theta})n = g_{n_\theta} g_{\theta}^{-1} \hat{N},
\]

(1.10)

where \( \hat{N} \) was initially along the z axis of the system frame located in \( K \) (i.e., parallel to \( s \)). If we consider \( \hat{N}_s \) and \( \hat{N}_S \), then

\[
\hat{N}_s = g_{s_1} g_{s_2}^{-1} \hat{N} = (g_{s_1} g_{s_2}) (g_{s_2} g_{s_1})^{-1} \hat{N},
\]

(1.11)

In the above, \( g_{\theta} \) was introduced as the rotation by \( \theta \) about the spatial axis. Each element of the rotation group can be represented that way. Let \( g_{\alpha} \) refer to the rotation axis \( \hat{e}_\alpha \) and angle \( \alpha \). By passing to \( \mu_j = \hat{e}_\mu \tan \alpha \), we arrive at a particularly convenient representation of spatial rotations by \( 3 \times 3 \) matrices \( R(\mu) \):

\[
R(\mu)_{ij} = [1/(1 + \mu^2)] [(1 - \mu^2) \delta_{ij} + 2 \mu \mu_j - 2 \epsilon_{ijk} \mu_k],
\]

(1.12)

with the composition rule

\[
R(\mu') R(\mu) = R(\mu''),
\]

(1.13)

allowing us to attribute to each rotation \( g \) in \( K' \) a respective spatial rotation \( \hat{g} \) in \( K \),

\[
g_{\theta} \xrightarrow{\text{g}} \hat{g}_{\theta} \xrightarrow{\text{g}} \hat{S}_{\theta} = (g_{\theta_\theta}) \hat{S}_{\theta} = \hat{g}_{\theta} \hat{S}_{\theta}.
\]

(1.14)

Notice that, together with \( S \), we have automatically defined an orthogonal reference triad in \( K \). The right screw
convention for the vector product allows us to introduce the analogs of the x and y axes as Sxv and (Sxv) xS, respectively.

All rotations can be parametrized by Euler angles introduced in this frame. Respective parametrizations are not the same for \{g_0^{-1}\} and \{g\}. However, the very concept of the invariant integration on S^3 implies that a given g_0 displacement on a group g \rightarrow g_0 does not affect the integration formulas, hence the respective Euler parameterization. We have \(dg=g_0dg\) and \(ff(g)dg=g_0ff(g_0)dg\). Consequently, the effective image of the group of rotations in \(K\) under the Lorentz transformation \(\Gamma\) is

\[
\mathcal{G}' \ni g \rightarrow \mathcal{G} = g'g_0',
\]

which well fits with (1.11):

\[
g_1'=g_1g_0^{-1}(g_0g_1) = \hat{\mathcal{N}}_1 = g_1g_0^{-1}(g_1'g_0'),
\]

If now the g's represent random rotations about the s polarized frame in \(K\), then \(g\)'s are their images as random rotations about the S polarized frame in \(K\).

Irrespective of whether we refer to \(K\)' or \(K\), the previous discussion and arguments of Ref. 3, Sec. IV tell us that once we have fixed the polarization axis direction (i.e., the z axis of the orthonormal triad), then the induced Euler angle parametrization of \(S^3\) allows us to determine the spin-\(\frac{1}{2}\) SU(2) harmonics as eigenfunctions of the Laplace–Beltrami operator on \(S^3\),

\[
\mathfrak{s} \rightarrow g = (\theta, \phi, \psi) \rightarrow \Delta_{\mathfrak{s}}f'(g) = \frac{1}{2}f'(g),
\]

\[
S \rightarrow g = (\theta, \phi, \psi) \rightarrow \Delta_Sf'(g) = \frac{1}{2}f'(g),
\]

\[
\Delta_S = \Delta_{(g \rightarrow g)},
\]

\[
\Delta_S = \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{\sin \theta} \frac{\partial}{\partial \phi} + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2} \right)
\]

\[
-\frac{2}{\sin \theta} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \psi},
\]

so that solutions acquire the same functional form, albeit with respect to entirely different parametrizations, thereby showing up different polarization directions in \(R^3\).

The change of the local \(S^3\) parametrization from \((\theta, \phi, \psi)\) to \((\theta', \phi', \psi')\) implies a replacement of the angular momentum vector \(L(g)\) by \(\bar{L}(g)\),

\[
L(g) = \alpha(g) e_\phi + \beta(g) e_\phi + \gamma(g) e_\phi,
\]

\[
\bar{L}(g) = \alpha(\bar{g}) e_\bar{\phi} + \beta(\bar{g}) e_\bar{\phi} + \gamma(\bar{g}) e_\bar{\phi},
\]

where the vectors \(e\) indicate directions about which rotations by the respective angles are executed. Since the \(e'\)'s are defined in the system frame (with the z axis given either by s or S), the transformation from \(e_\phi\), \(e_\phi\), \(e_\phi\) to \(e_\bar{\phi}\), \(e_\bar{\phi}\), \(e_\bar{\phi}\) is effected by the previously considered spatial rotation \(g_0\), taking s into S. Accordingly,

\[
\bar{L}(g) = (g_0L)(g \rightarrow g),
\]

i.e., the change of arguments is accompanied by the overall rotation of L. We then have

\[
\langle \bar{L} \rangle = \frac{1}{2} \int L(g) |e_\phi(g)|^2 dg = \frac{1}{2} \int L(g) |e_\phi(g)|^2 dg.
\]

Consequently, four stationary states of rotational diffusion, \(e_\phi(g), g = (\theta, \phi, \psi)\) in \(K\), can be mapped into four states of rotational diffusion again, while their polarization is taken over to \(S\).

This map we shall study in more detail in connection with solutions of the Dirac equation.

II. ROTATIONAL DIFFUSIONS AS SEEN BY A RELATIVISTIC OBSERVER: CASE OF UNIFORM MOTION

The description of a stochastic process is usually confined to a fixed time interval, which eventually might be extended to an arbitrary size. Let us choose \([0, T]\) \(\Rightarrow t\). A random variable \(G(t')\) is represented by a rotational event \(g\) taking place at time \(t'\), while \(x\) is the location of the origin of the rotating triad. Hence we deal with \(g\) at the space-time point \((ct', x')\). By virtue of our previous considerations, \(G(t')\) induces a random variable \(G(t')\) in \(K\) that refers to a rotation \(\hat{g}\) taking place at the space-time point \((ct, x)\) in \(K\),

\[
x = x' + v[(\gamma - 1)(ux)/v^2 + yt],
\]

\[
t = t' + (ux/v^2).
\]

Since \(x'\) is fixed and the time label \(t'\) is only allowed to vary, we can rewrite (1.1) as \(x = x_0 + y\gamma t', t = t_0 + t'\). It tells us that the process is perceived in \(K\) as taking place in the time interval \([\gamma t_0, \gamma (t_0 + T)]\) while the rotating triad origin is propagated uniformly with velocity \(v\) from the spatial location \(x = x_0\) to \(x = x_0 + \gamma vT\).

Here \(x_0\) is associated with the origin of the rotating frame; hence it is a valid assumption to consider \(x_0 = 0\) only. This yields
If \( u \) is the speed of the particle relative to the reference
frame in which the time interval is measured, the notion
of proper time comes from \( \Delta r = \Delta t/\gamma \).

Apparently it gives rise to a proper time labeling of
random variables,

\[
K' \rightarrow K \Rightarrow G(t') \rightarrow \bar{G}(t') = \bar{G}(t/\gamma) = \bar{G}(\tau),
\]

under the assumption (2.2).

Consider a small surface on \( S_3 \) with the area \( \Delta g \) cen-
tered about the point (rotation) \( g \). Let \( \epsilon_i(g) \) be the \( i \)th
state of rotational diffusion. Then \( \phi_i(g) \Delta g \) represents the probability with which rotations close to \( g \) are met
along sample paths in the infinite sampling limit: then the
frequency of an event approaches a probability of its oc-
currence.

**Remark I:** The notion of randomness automatically
induces the notion of sampling: a repeatable processing
confined to a fixed time interval. In particular, the notion
of sample paths\(^3\) of a given stochastic process is of pro-
found importance. In \( K' \) it amounts to representing
the random propagation on \( S_3 \) by a collection of random trajecto-
ries: they are different realizations (samples) of a
given random motion scenario in the time interval \([0, T']\),
executed by the random variable \( G(t') \).

The same process, but seen from another inertial
frame \( K \), induces sample paths as rotational events, which
are continuously distributed along the relativistic path.

The transformation \( A \) affects only the longitudinal
component \( x_i \) of \( x' \),

\[
i\hbar \delta f(g', \tau) = (-\hbar^2/2I) \Delta g f(g', t'),
\]

where we set \( I = 3\hbar^2/8mc^2 \) to deal with spin\(^{1/2} \). The re-
spective eigenvalue problem in \( K \) is given by the Schrö-
dinger equation on \( S_3 \),

\[
f(g', t') = f(g, t') \exp(-mc^2t'/\hbar),
\]

\[
(\tau = \gamma(t' - \frac{v}{c} \frac{\xi}{v}), \quad \xi = \frac{v}{u} |p| x \),
\]

while on the other hand, \( p_{\mu} \xi^{\mu} \) emerges by setting \( p = \gamma mu. \)

The formula (2.9) maps the \( x_i \) plane in \( K' \) into the \( x_i \)
plane in \( K \). The space-time location of the plane in \( K \) is
uniquely defined by a corresponding time instant \( t' \) of the
rest frame evolution. By (2.9) the one-parameter family
of wave fronts at rest (in \( K' \)) is perceived in \( K \) as the
one-parameter family of traveling surface (planes):

\[
f(G, \tau) = f(G) \exp(-iE \tau/\hbar),\]

which is characteristic of plane wave solutions of the
Dirac equation, except for the explicit \( \frac{\mu}{\hbar} \) dependence of the
coefficient \( f(G) \).

At this point it is quite instructive to invoke an ex-
haustive discussion of Ref. 4 on the determination of the
rest mass and spin of the particle in the context of rela-
tivistic invariant wave equations.

Usually one deals with arbitrary plane wave solutions
and attempts to extract their rest frame properties. We
have proceeded in reverse, while having a detailed rest
frame picture (of random phenomena) in hand. Let us
view (2.7) as an arbitrary plane wave, i.e., allow \( t \) and \( x \)
to take any value. We can always pass\(^4\) to the rest frame
of the wave and recover a corresponding stationary plane
wave, which is (2.5) in our case. Indeed, the Lorentz
transformation \( A : (\gamma c, \gamma u, x') \rightarrow (c, 0, 0, 0) \) implies

\[
f(G) \exp(-i\eta(x' \hbar/\hbar)) \rightarrow f'(g) \exp(-i\eta(\hbar/c)^2t'/\hbar),
\]

for all \( x \) and \( t \). Although the spatial image \( x' \) of \( x \) under
\( \Lambda \) is not manifestly present in (2.8), it is implicitly there,
since in our framework spatial rotations \( g \) take place as
rotations about this point.

The transformation \( \Lambda \) affects only the longitudinal
component \( x_1 \) of \( x' \),

\[
x_1 = \gamma(x_1' + \eta t'), \quad \gamma = \frac{v}{u} |p| x \)
\]
and not at all to what we usually call traveling waves (Ref. 10 addresses the issue in more detail).

III. DIRAC EQUATION IN STOCHASTIC MECHANICS:
REVIVAL OF SOME OLD IDEAS

The stochastic implementation of the quantum spin-\(\frac{1}{2}\) system involves four distinct states of rotational diffusion, which reflect the existence of left and right representations of the SU(2) on \(\mathcal{H}_{1/2}\). In \(K'\) we have

\[
(8\pi^2)^{1/2}e_1(g) = i \cos \frac{\theta}{2} \exp i \frac{\theta}{2}(\psi + \phi) \sim d_{1/2}^{1/2}(g),
\]

\[
\langle L \rangle_1 = -\frac{\hbar}{2} k,
\]

\[
(8\pi^2)^{1/2}e_2(g) = i \sin \frac{\theta}{2} \exp \frac{1}{2}(\phi - \psi) \sim d_{1/2}^{1/2}(-1)(g),
\]

\[
\langle L \rangle_2 = \frac{\hbar}{2} k,
\]

\[
(8\pi^2)^{1/2}e_3(g) = i \sin \frac{\theta}{2} \exp i \frac{1}{2}(-\phi + \psi) \sim d_{1/2}^{1/2}(-1)(g),
\]

\[
\langle L \rangle_3 = -\frac{\hbar}{2} k,
\]

\[
(8\pi^2)^{1/2}e_4(g) = i \cos \frac{\theta}{2} \exp i \frac{1}{2}(-\phi - \psi) \sim d_{1/2}^{1/2}(-1)(g),
\]

\[
\langle L \rangle_4 = -\frac{\hbar}{2} k,
\]

where \(d_{1/2}^{1/2}\) is the standard notation for SU(2) harmonics. The respective stochastic processes are determined by computing the angular velocity \(\omega(g)\) induced by the rotation \(g\) and \(\omega(g)\) is a sum of the current \(\omega_c\) and osmotic \(\omega_o\) contributions behaving differently under time reversal. Namely, \(t' \rightarrow -t'\) implies \(\omega_o \rightarrow -\omega_o\) while \(\omega_c \rightarrow -\omega_c\).

As a consequence (compare, e.g., Sec. IV of Ref. 2), we arrive at

\[
\begin{align*}
\omega^1_v &\rightarrow \omega^4_v, & \omega^2_v &\rightarrow \omega^3_v, \\
\omega^1_u &\rightarrow \omega^4_u, & \omega^2_u &\rightarrow \omega^3_u
\end{align*}
\]

which amounts to the map

\[
e_1(g) \rightarrow e_4(g), \quad e_2(g) \rightarrow e_3(g), \quad e_3(g) \rightarrow e_2(g), \quad e_4(g) \rightarrow e_1(g),
\]

\[
f'(g,t') \rightarrow f'(g,-t') = e(g)\exp(mc^2t'/\hbar).
\]

Remark 2: Four states of rotational diffusion (3.1) were introduced in connection with the forward propagation. Apparently, the discussion of how to describe effects of time reversal in stochastic mechanics as a forward propagation again may be adopted here. Usually the reverse process is viewed as the random propagation in the backward direction, which allows one to reproduce past (statistical) data of the process given the present, hence as a mere mathematical artifice. It appears that in the case of spin-\(\frac{1}{2}\) diffusions it is no longer so. The arguments of Ref. 14, the Introduction, tell us that for Markovian diffusions we can define a forward process that is the exact time reversal of another forward process, and the diffusions underlying (3.1) provide us with explicit examples.

Let us recall-\(1^3\) that the SU(2) labeling of eigenfunctions (3.1) is provided by the eigenvalues of the operators \(M_3, N_3\), where \(M\) is the generator of left rotations while \(N\) is the (abnormal) generator of right rotations. We have \(M^2 = N^2 = -\hbar^2\Delta_s\) on the S3 manifold, and \(M_3 = -i\hbar d/\partial\phi, N_3 = -i\hbar d/\partial\psi\). The eigenvalues of \(M_3\) correspond to expectation values \(\langle L \rangle_1\) of the angular momentum (spin) arising due to the rotational diffusion.

The ordering \((e_1,e_2,e_3,e_4)\) of the basis system refers to a \((+,+,-,-)\) sequence of the \(M_3\) eigenvalues and to \((+,+,-,-)\) for \(N_3\). Analogously, \((e_2,e_4,e_1,e_3)\) refers to \((+,-,+,-)\) for \(M_3\) and \((-,-,+,-)\) for \(N_3\). In view of this, formulas (2.8) and (3.3) give rise to two distinct evolution equations in \(K'\) that encompass the time reversal in a manifest way. Namely, \(e_j(g)\exp(-imc^2t'/\hbar), j = 1,3\) and \(e_k(g)\exp(imc^2t'/\hbar), k = 2,4\) form a set of independent solutions of the equations

\[
i\hbar \partial_t f'(g,t') = -\langle 2/\hbar \rangle mc^2N_3f'(g,t'),
\]

while \(e_j(g)\exp(-imc^2t'/\hbar), k = 2,4\) and \(e_k(g)\exp(imc^2t'/\hbar), j = 1,3\) for

\[
i\hbar \partial_t f'(g,t') = \langle 2/\hbar \rangle mc^2N_3f'(g,t').
\]

The “positive energy” solutions of (3.4) and (3.5) constitute the orthonormal set in \(\mathcal{H}_{1/2}\). The prime refers to the rest frame Euler parametrization.

Remark 3: The above observation, if combined with the previous Remark 2, lends weight to Barut’s conjecture \(1^5\) that perhaps there is no real need to invoke the hole theory or the notion of backward propagation in time to describe antiparticles.

Let us address the question of how the rest frame evolution (respectively, the eigenvalue problem for \(N_3\)) equations (3.4) and (3.5) are seen in another Lorentz frame.

In accordance with the standard rules of the game the Lorentz transformation \(A: K \rightarrow K'\) should imply a nonunitary map in the function space, replacing the \(K\) frame data by the \(K'\) ones,

\[
f'(x',g) = (T_Af)(x',\bar{g}).
\]
We shall investigate the outcome of (3.6) in the Hilbert space spanned by $e_i(\vec{g})$ with the $L^2(S_3)$ scalar product $f_1(\vec{g})f_2(\vec{g})d(\vec{g})=(f_1,f_2)$ valid in $\mathcal{K}$. Let us consider a transformation,

$$\Lambda: \mathcal{K} \rightarrow \mathcal{K}' \Rightarrow f(\vec{g}) = (T_Af)(\vec{g}), \quad (3.7)$$

where $T_\Lambda = T_\Lambda(M,N)$ is given by (this formula was first introduced to represent Lorentz transformations in Ref. 13)

$$T_\Lambda = \cosh \frac{\eta}{2} \left[ c + \frac{1}{E+mc^2} \left( \frac{2}{\eta} \right)^2 N_1(pM) \right], \quad (3.8)$$

with

$$\eta = \frac{E+m^2c^4}{2mc^2}^{1/2}, \quad p^0 = \frac{E}{c} = (p^2+m^2c^2)^{1/2}. \quad (3.9)$$

In the above, $N$ and $M$ are the previously defined differential operators on $S_3$ whose explicit form displays the local $(\theta, \phi)$ parametrization.

By exploiting the formulas valid for an irreducible representation of the group of rotations in $L^2(S_3)$,

$$(M_1+iM_2)d_{mn}^{l/2} = \eta [ (s+m)(s+m+1) ]^{1/2} d_{mn-1,1}^{l/2}, \quad (3.10)$$

and specializing them to spin-$1/2$, we obtain

$$N_1d_{1/2}^{1/2} = (\eta/2)d_{1/2}^{1/2}, \quad M_2d_{1/2}^{1/2} = -i(\eta/2)d_{1/2}^{1/2}, \quad (3.11)$$

$$N_1d_{1/2}^{1/2} = (\eta/2)d_{1/2}^{1/2}, \quad M_2d_{1/2}^{1/2} = -i(\eta/2)d_{1/2}^{1/2}.$$ 

This entails an immediate evaluation of the action of $T_\Lambda$ on any of the $e_i$'s. Let us introduce the notation

$$\{e_1,e_3,e_2,e_4\} = \{\phi_1,\phi_2,\phi_3,\phi_4\}. \quad (3.12)$$

Then we arrive at

$$(T_\Lambda \phi_i)(\vec{g}) = \sum_{k=1}^{4} \phi_k(\vec{g}) = \sum_{k=1}^{4} S^T_{ik} \phi_k, \quad (3.13)$$

where one recognizes $S^T$ to be a transposed bispinor transformation matrix [see (3.7) in Ref. 16],

$$S = \exp \left( -\frac{\eta}{2} \frac{\alpha u}{|u|} \right) \gamma' = \gamma^0 \alpha_i \quad (3.14)$$

$$\psi'(x') = \psi(Ax) = S(A) \psi(x) = S(A) \psi(A^{-1}x').$$

Accordingly, the $4 \times 4$ matrix $S$ comes out by evaluating matrix elements of the operator $T_\Lambda$ in the $\{\phi_i(\vec{g})\}$ rotational basis.

Remark 4: Let us emphasize that our analysis is carried out in a four-dimensional vector space, which is a natural module for the compact group $SU(2)_L \times SU(2)_R$. In order to establish a relativistic description (rather to exploit what is known about the relativistic covariance of the Dirac equation) the same space is required to act as a module for the noncompact group $SL(2,C)$. The latter action does not seem to arise that naturally, except for rather conspicuous affinity (dimension four) with the standard bispinor transformations induced by the Lorentz mapping. In fact, $T_\Lambda$ (3.8), in view of its irredicible action on the four-dimensional carrier space, is equivalent (matrix form!) to the well-known mappings: (3.13) and (3.14) should be compared with the formula (3.7) of Ref. 16. Our procedure should not be confused with the general $SO(4)$ complexification problem. In fact, this point makes the original Dahl's proposal indigestible: the $SL(2,C)$ covariance cannot be naively replaced by the $SO(4)$ covariance. Although the $SO(4)$ covariant $(SU(2) \times SU(2)/Z_2)$ spin-$\frac{1}{2}$ system is our starting point, we pass to a new $SL(2,C)$ covariant spin-$\frac{1}{2}$ system built on the carrier (representation) space of the former. This task is accomplished by means of the nonunitary representation of $SL(2,C)$ for which finite-dimensional realizations are known to exist. The role of $M,N$ generators is different in the $SL(2,C)$ case if compared with $SO(4)$. One of them, instead of generating rotations, gives rise to Lorentz boosts. This feature is completely revealed by formulas (3.8)-(3.14).

Let $w(r)(\vec{p})$ be the $r$th column of the matrix $S$. We can rewrite (3.13) as follows:

$$(T_\Lambda \phi_r)(\vec{g}) = \sum_{k=1}^{4} w_k(\vec{p}) \phi_k(\vec{g}) = \phi_r(\vec{p},\vec{g}). \quad (3.15)$$

The $L^2(S_3)$ orthonormality relations imply here (we use the bispinor normalization identity in the second step)

$$(\phi_r^*,(\epsilon,\vec{p}),\phi_{r'}^*(\epsilon,\vec{p})) = w^{*r}(\epsilon,\vec{p})w(\epsilon,\vec{p}) = \frac{E}{mc^2} \delta_{rr'}, \quad (3.16)$$

$$\epsilon_r = \pm 1, \quad r = 1, 2, \quad \epsilon_r = -1, \quad r = 3, 4,$$

and allow us to introduce a new orthonormal basis system in $\mathcal{H}_{1/2}$ encompassing the effects of the Lorentz transformation $\mathcal{K} \rightarrow \mathcal{K}'$, 

\[ e'_1(\vec{g}) = \phi'_1(\vec{p}, \vec{g}) \left( \frac{m^2}{E} \right)^{1/2}, \quad e'_3(\vec{g}) = \phi'_2(\vec{p}, \vec{g}) \left( \frac{m^2}{E} \right)^{1/2}, \]

\[ e'_2(\vec{g}) = \phi'_3(-\vec{p}, \vec{g}) \left( \frac{m^2}{E} \right)^{1/2}, \quad (3.17) \]

\[ e'_4(\vec{g}) = \phi'_4(-\vec{p}, \vec{g}) \left( \frac{m^2}{E} \right)^{1/2}, \]

where \((e'_i, e'_j) = \int d\vec{g} e'_i(\vec{g}) e'_j(\vec{g}) = \delta_{ij}\) is a positive definite sesquilinear form.

More explicitly,

\[ e'_1(\vec{g}) = \left( \frac{L + m^2}{2E} \right)^{1/2} \left[ e_1(\vec{g}) + \frac{c}{E + mc^2} \right] \times \left[ p, e_2(\vec{g}) + p, e_4(\vec{g}) \right], \]

\[ e'_2(\vec{g}) = \left( \frac{L + m^2}{2E} \right)^{1/2} \left[ e_2(\vec{g}) + \frac{c}{E + mc^2} \right] \times \left[ p, e_1(\vec{g}) - p, e_4(\vec{g}) \right], \quad (3.18) \]

\[ e'_3(\vec{g}) = \left( \frac{L + m^2}{2E} \right)^{1/2} \left[ e_3(\vec{g}) + \frac{c}{E + mc^2} \right] \times \left[ -p, e_1(\vec{g}) - p, e_2(\vec{g}) \right], \]

\[ e'_4(\vec{g}) = \left( \frac{L + m^2}{2E} \right)^{1/2} \left[ e_4(\vec{g}) + \frac{c}{E + mc^2} \right] \times \left[ -p, e_1(\vec{g}) + p, e_2(\vec{g}) \right], \]

\[ \rho = (p_x, p_y, p_z), \quad \rho_\pm = p_x \pm ip_y, \]

where \(e'_i(\vec{g})\) have the form (3.1), except for the replacement of \(\theta, \phi, \psi\) by \(\tilde{\theta}, \tilde{\phi}, \tilde{\psi}\).

It was demonstrated for the first time in Ref. 13 that the functions

\[ f_j(\vec{g}, x) = (T_\Lambda \phi_j) \exp(-im \frac{\epsilon_j x^j}{\hbar}), \quad (3.19) \]

solve the evolution equation

\[ i\hbar \partial_t f_j(\vec{g}, x) = \left[ \frac{2}{\hbar} m^2 N_3 + \frac{4c}{\hbar} N_1 [M(-i\hbar \nabla)] \right] f_j(\vec{g}, x). \quad (3.20) \]

Its matrix form in the \(\{\phi_j(\vec{g})\}\) basis is the familiar Dirac equation

\[ \{mc^2\psi + \alpha(-i\hbar \nabla)\} \psi = i\hbar \partial_t \psi. \quad (3.21) \]

The image of (3.5) under \(\Lambda\) is obtained through replacing \(m\) by \(-m\) in the above.

The equation (3.21) is known to be Lorentz invariant. Then what about (3.20)? By setting

\[ L^0 = I, \quad L^i = -N_1 M_{ip}, \quad i = 1, 2, 3, \quad (3.22) \]

(3.20) can be cast in the manifestly covariant form

\[ \frac{2}{\hbar} m^2 N_3 f_j = \alpha^\mu \partial_{\alpha} f_j. \quad (3.23) \]

The standard Lorentz covariance arguments require that (3.6) be a map of the \(K\) frame data into the \(K'\) frame ones. Accordingly,

\[ N_3 - T_\Lambda N_3^{-1} T_\Lambda^{-1}, \quad (3.24) \]

reflects merely the change of the Euler parameters from \(\tilde{\theta}, \tilde{\phi}, \tilde{\psi}\) to \(\theta, \phi, \psi\) as a result of the Lorentz transformation, while there holds

\[ T_\Lambda L^\mu T_\Lambda^{-1} a_\mu = L^\nu \partial_\nu f_j(\vec{g}, x'), \quad (3.25) \]

\[ \alpha^\mu = \partial_\nu \psi. \]

The \(K'\) frame version of (3.20),

\[ \frac{2}{\hbar} m^2 N_3 f_j^*(g, x') = \alpha^\nu \partial_\nu f_j^*(g, x'), \quad (3.26) \]

reduces to (3.4) in the case of plane wave solutions.

As a consequence of (3.6), we realize that stationary plane wave solutions \(\phi_j(\vec{g}) \exp(-im \epsilon_j x^j/\hbar)\) of (3.4) are represented in terms of the \(K\) frame data by the solutions (3.17) of the evolution equation (3.20). A serious problem comes here from the covariant normalization statement

\[ \tilde{w}^*(p) w' = \delta_{\rho \sigma} e_\sigma. \quad (3.27) \]

Before, the plane waves were found to refer to four distinct stochastic rotational processes in \(K'\). Because of the improper (negative) normalization, the \(r = 3\) and 4 images of random motions in \(K'\) do not admit any reasonable probabilistic meaning in \(K\), hence they cannot be perceived as stochastic processes in \(K\). It is the normalization identity (3.15) that allows us to introduce an orthonormal basis system (3.16) with prospects for a correct probabilistic content (due to a positive normalization). Apparently (3.16) arises only if we consider a complete set of “positive energy” solutions of both (3.4) and (3.5). Both these evolution equations are in-
dispensable for a covariant transformation of the orthonormal basis given in $K'$ into an orthonormal basis in $K$. In fact, the formulas (3.16) identify these functions in $\mathbb{R}^3/t'$, which provide us with the $K$ frame image (via Lorentz transformation) of four distinct stochastic processes in $K'$. This map allows us to perceive certain $K'$ frame diffusion as genuine diffusion processes in the inertial frame $K$.

Since all $e_i(g)$ solve the eigenvalue problem $\Delta g' e_i(g) = \frac{1}{2} e_i(g)$, the basis functions (3.16) solve it as well. A complete stochastic decoding of (3.16) amounts to a repetition of Dankel's strategy once the $e_i's$ are cast in the canonical (Madelung) form $e' = \exp(R + is)$. Now $|e'_i(g)|^2$ represents the probability distribution of the $i$th stationary diffusion as perceived in $K$. The respective random variable is labeled by the proper time.

We may now formulate a definite answer to the question raised in the Introduction. What is perceived in $K$ as a stochastic rotational diffusion is no longer a diffusion associated with the forward time development exclusively, i.e., $e_i(g)\exp(-imc^2t'/\hbar)$ for all $i=1,2,3,4$. The answer is positive if we go over to the rest frame diffusions associated with the evolutions $e_j(g)\exp(-imc^2t'/\hbar)$ for $j=1,3$, and $e_k(g)\exp(imc^2t'/\hbar)$ for $k=2,4$. The backward evolution for $k=2,4$ can be represented as a forward evolution again by invoking the arguments of Ref. 14, but is irreducibly different from the one associated with $e_k(g)\exp(-imc^2t'/\hbar)$.

Our analysis allows one to associate diffusions on $S_3$ with plane wave solutions of the Dirac equation, which is possible due to the implicit validity of the proper time Schrödinger equation on $S_3$. We then deal with rotational fluctuations that are intrinsic to a particle in uniform motion. There is no essential difficulty in extending the arguments to cases covered by the semiclassical regime for solutions of the Dirac equation in the presence of external electromagnetic fields (inhomogeneities included). The proper time evolution governed by the Bargmann–Michel–Telegdi equation amounts to a purely rotational diffusion process, which is effected along a space-time trajectory of the particle. The motions show the same feature: randomness is exclusively intrinsic and does not affect the space-time path followed by the origin of the rotating frame (on the contrary, it is rather that the spin precession is strongly path dependent.

The problem we have left aside at the moment is the probabilistic analysis of general wave packet solutions of the Dirac equation, where a nontrivial input of the random process affecting a particle velocity (extrinsic randomness) is expected to show up.

Since random paths of stochastic mechanics are quite akin to Feynman paths, it should, in principle, be possible to establish a unifying framework for an increasing number of path integral approaches to the description of Dirac particles in the non-Grassmann vein. It especially pertains to random walk representations of the Dirac propagator where one generally assumes that at each step of the random walk executed by the spinning particle in Minkowski space, its quantization axis is rotated by a certain angle. Compare, e.g., our discussion of the Introduction, where momentum change induces a well-defined rotation of the polarization.

References to numerous relativization attempts in the context of Nelson's stochastic mechanics can be found in the recent papers; also see Refs. 25 and 26. A problem worth a deeper exploration in the presented probabilistic framework is the physical meaning of different notions of position invented for the Dirac particle, and also of Zitterbewegung, which from our perspective is definitely not the intrinsic mechanism implementing the electron spin. On the other hand, the recent magnetic top model, albeit devoid of any explicit randomness, shows up all basic features discussed in the present paper.

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