Cauchy semigroups: Nonlocally-induced bound states

(related to nonlocal random motions and their equilibria)

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The generator $\Delta = \partial^2/\partial x^2$ of the standard Brownian motion in R , **is locally defined**

To the contrary , there exists a plethora of **nonlocal** generators (and related **nonlocal random processes)**. Typical examples:

$-|\Delta|^{\mu/2}$ $\mu \in (0,2)$ Lèvy – stable driver

$\sqrt{-\hbar^2 c^2 \Delta + m^2 c^4} - mc^2$ quasi-relativistic process, **for m=0 Cauchy process**

$$-|\Delta|^{\mu/2} f(x) = \int [f(x+y) - f(x)]\, \nu_\mu(dy) = \frac{2^\mu\, \Gamma(\frac{\mu+1}{2})}{\pi^{n/2} |\Gamma(-\frac{\mu}{2})|} \int \frac{f(y) - f(x)}{|x-y|^{\mu+n}}\, dy$$

the Cauchy principal value

here $x \in R^n$

Quiery: How technically can we **confine** the (exemplary, **nonlocal**) Cauchy process in a finite box (e.g. interval [-1,1]) ? What is a nonlocality impact on the **approach to equilibrium** and the asymptotic pdf **shape** in a finite trap ?

Methodology: semigroup versus Feller dynamics

(i) **Semigroup** input, Levy processes with **killing**, transition kernels , „spectral properties of stochastic processes", eigenfunction expansions, spectral solutions for involved motion generators .

(ii) We explore **an intrinsic connection** with standard **Feller processes**: those are **without killing** and do **respect** so-called natural boundaries (effectively inaccessible from the trap interior) .

Hint: Depart from the standard problem of the Brownian motion in a box and try to answer **what is actually meant by the Brownian motion in a trap**. Next, address the problem: is there anything similiar in case of the **Cauchy** process ?

(notes borrowed from the talk by M. Kwasnicki, IMath Wrocław Univ. Techn.)

Setting (1/3)

- X_t is always a **Lévy process** in $\mathbf{R}$
- $\mathbf{P}_x$, $\mathbf{E}_x$ correspond to $X_0 = x$
- We assume that X_t is **symmetric**

Lévy-Khintchine

- Ψ is the **Lévy-Khintchine exponent**:

$$\mathbf{E}_0 \exp(i\xi X_t) = \exp(-t\Psi(\xi))$$

$$\Psi(\xi) = \beta\xi^2 + \int_{-\infty}^{\infty} (1 - \cos(\xi z))\nu(dz)$$

- β is the **diffusion coefficient**
- ν is the **Lévy measure**

Free noise; keep the heat kernel notion in mind (Brownian association)

Setting (2/3)

- Transition density:

$$p_t(x, y)dy = \mathbf{P}_x(X_t \in dy)$$

- Transition operators:

$$P_t f(x) = \mathbf{E}_x f(X_t) = \int_{-\infty}^{\infty} p_t(x, y) f(y) dy$$

- Generator:

$$\mathcal{A}f = \lim_{t \to 0^+} \frac{P_t f - f}{t}$$

- Lévy-Khintchine:

$$\mathcal{F}\mathcal{A}f(\xi) = -\Psi(\xi)\mathcal{F}f(\xi)$$

The validity of the Fourier multiplier picture is presumed

Restriction to a finite domain D: transition densities no longer integrate to 1 .

Setting (3/3)

- First exit time:

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\}$$

- Killed process:

$$X_t^D = \begin{cases} X_t & \text{when } t < \tau_D \\ \partial & \text{when } t \geq \tau_D \end{cases}$$

- Transition density:

$$p_t^D(x,y)dy = \mathbf{P}_x(X_t^D \in dy) = \mathbf{P}_x(X_t \in dy; t < \tau_D)$$

- Transition operators:

$$P_t^D f(x) = \mathbf{E}_x f(X_t^D) = \int_D p_t^D(x,y) f(y) dy$$

- Generator:

$$\mathcal{A}_D f = \lim_{t \to 0^+} \frac{P_t^D f - f}{t}$$

Stochastic processes with **exterior Dirichlet boundary condition,** may be interpreted in terms of the **semigroup dynamics** with a singular (**infinite well** – type) potential as an additive perturbation.

We can always regularize that problem by passing to a family of monotonically deepening **finite well** problems and ask for a **deviation** of a **very deep well** spectral solution from spectral data of the **infinite well .**

Math. issue: self-adjointness of the generator in the well.

That is about: *H = T +V,* $T_0 = \hbar c|\nabla|$ Cauchy generator equals -T

Confining V=V(x): **harmonic** or **finite well** of arbitrary depth

- **Lévy-Schrödinger semigroups** - additive perturbations of nonlocal noise generators, spectral properties

Spectral solution of the semigroup operator is instrumental for both an identification of an equilibrium pdf (square of the normalized lowest eigenfunction) and of the dynamics details of the related Feller process.

Eigenfunctions and eigenvalues of exp(-tH) fully determine the equilibrating random motion and in particular its near equilibrium behavior .

Why possibly spectral solutions may be useful ? Answer: good approximate formulas, control of the asymptotic behavior, known spectral gaps, if in existence.

Eigenfunction expansions for the free Brownian motion – heat kernel issue

For clarity of discussion, it is instructive to invoke explicit examples. We pass to one spatial dimension and rescale (or completely scale away) a diffusion coefficient. Given a spectral solution for $\hat{H} = -\Delta + V \geq 0$ in $L^2(R)$, the integral kernel of $\exp(-t\hat{H})$ reads ($t \to it$ gives rise to the kernel of $\exp(-it\hat{H})$)

$$k(y, x, t) = k(x, y, t) = \sum_j \exp(-\epsilon_j t)\, \Phi_j(y)\Phi_j^*(x). \tag{16}$$

Remember that we assume $\epsilon_0 = 0$ and the sum may be replaced by an integral in case of a continuous spectrum. Then one needs to employ complex-valued generalized eigenfunctions, e.g., $\Phi_j(x) \to \Phi_p(x) = (2\pi)^{-1} \exp(ipx)$. Indeed, if we set $V(x) = 0$ identically, a familiar heat kernel is readily obtained

$$k(y, x, t) = [\exp(t\Delta)](y, x) = \frac{1}{2\pi} \int \exp(-p^2 t) \exp(ip(y - x)\, dp = \tag{17}$$

$$(4\pi t)^{-1/2} \exp[-(y - x)^2/4t]\,,$$

in accordance with $\int \exp(-\sigma^2 p^2) \exp(-ipx) dx = (\pi/\sigma)^{1/2} \exp(-p^2/4\sigma^2)$, where $\sigma > 0$. We note that the kernel of $[\exp(tD\Delta)]$ appears after changing the time scale in (17), $t \to Dt$. A formal identification $D \equiv 1/2$ gives the kernel that is often met in the mathematical literature and corresponds to $(1/2)\Delta$ instead of Δ.

The Ornstein-Uhlenbeck process: semigroup vs Feller motion scenario

Consider $\hat{H} = (1/2)(-\Delta + x^2 - 1)$ (e.g., the rescaled and (-1) renormalized harmonic oscillator Hamiltonian). The integral kernel of $\exp(-t\hat{H})$ is given by a rescaled form of the classic Mehler formula:[15,16]

$$k(y,x,t) = [\exp(-t\hat{H})(y,x) = \frac{1}{\sqrt{\pi}} \exp[-(x^2+y^2)/2] \sum_{n=0}^{\infty} \frac{1}{2^n n!} H_n(y) H_n(x) \exp(-nt) = \quad (18)$$

$$(\pi[1-\exp(-2t)])^{-1/2} \exp\left[-\frac{1}{2}(x^2-y^2) - \frac{(x-e^{-t}y)^2}{(1-e^{-2t})}\right],$$

where $\epsilon_n = n$, $\Phi_n(x) = [4^n(n!)^2\pi]^{-1/4}\exp(-x^2/2)H_n(x)$ is the $L^2(R)$ normalized Hermite (eigen)function, while $H_n(x)$ is the nth Hermite polynomial $H_n(x) = (-1)^n(\exp x^2)\frac{d^n}{dx^n}\exp(-x^2)$.

The normalization condition $\int k(y,x,t)\exp[(y^2 - x^2)/2]\, dy = 1$ actually defines a transition probability density of the Ornstein-Uhlenbeck process (see, e.g., Eq. (19))

$$p(y,0,x,t) \equiv p(y,x,t) = k(y,x,t)\, \rho_*^{1/2}(x)/\rho_*^{1/2}(y) \quad (19)$$

with $\rho_*(x) = \pi^{-1/2}\exp(-x^2)$.

A more familiar form of the Mehler kernel reads (note the presence of $\exp(t/2)$ factor)

$$k(y,x,t) = \frac{\exp(t/2)}{(2\pi \sinh t)^{1/2}} \exp\left[-\frac{(x^2+y^2)\cosh t - 2xy}{2\sinh t}\right]. \quad (20)$$

Brownian motion in the interval (-1,1): semigroup picture

The orthonormal eigenbasis is composed of functions $\psi_n(x)$, $n = 1, 2, \ldots$ such that $\psi(x) = 0$ for $|x| \leq a$, where n labels positive eigenvalues $E_n \sim n^2$. More explicitly: $\psi_n(x) = \cos(n\pi x/2)$ for n even and $\sin(n\pi x/2)$ for n odd, while the eigenvalues read $E_n = (n\pi/2)^2$.

It is clear that any $\psi \in L^2([-1,1])$, in the domain of the infinite well Hamiltonian, may be represented as $\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$. Its time evolution follows the Schrödinger semigroup pattern $\psi(x) \to \Psi(x,t) = [\exp(-Ht)\psi](x) = \sum_{n=1}^{\infty} c_n \exp(-E_n t)\, \psi_n(x)$.

Let us consider $H = -\Delta - E_1$ instead of $H = -\Delta$ proper (the boundary data being implicit). Accordingly, the *a priori* positive-definite ground state $\psi_1(x) \doteq \rho_*^{1/2}(x)$ corresponds to the zero eigenvalue of $H - E_1$. Thence, the "renormalized" semigroup evolution reads $\Psi(x,t) = \exp(+E_1 t) \sum_{n=1}^{\infty} c_n \exp(-E_n t)\, \psi_n(x) \to \psi_1(x) = \rho_*^{1/2}(x)$. Here, in a self-explanatory notation, we have defined the probability density function (pdf) $|\psi_1(x)|^2 = \rho_*(x)$

The semigroup kernel $\exp(-tH)(x,y)$, associated with such $\hat{H}$ whose lowest eigenvalue is 0, defines a time homogeneous random process in the interval. Its standard spectral representation is (the renormalization by $-E_1$ produces here an exponential factor), see also [9]

$$\begin{aligned} k(t,x,y) &= \exp(-Ht)(x,y) \\ &= \exp\left(+\pi^2 t/4\right) \sum_{n=1}^{\infty} \exp\left[-(n\pi/2)^2\, t\right]\, \psi_n(x)\, \psi_n(y) \\ &= \sum_{n=1}^{\infty} \exp\left[(1-n^2)\, \pi^2\, t/4\right] \sin[n\pi(x+1)/2]\, \sin[n\pi(y+1)/2]\,. \end{aligned}$$

Here, $\Psi(x,t) = \int k(t,x,y)\Psi_0(y)\, dy$. In probabilistic terms, the kernel allows to define a conditional probability $P_x(X_t) = k(t,x,y)dy$ that a process started at x will reach a vicinity dy of y in time t.

In the standard lore of the Brownian motion with killing (sometimes identified with absorption), one adds that t is prior to a killing time τ. An inventory of typical calculable functions/functionals related to the killed Brownian motion (various forms of the transition density k, distribution function and density of first exit time τ, mean first passage/exit time, *etc.*)

Schrödinger semigroup transcript of the Fokker-Plack dynamics.

The essence of the method (we consider the 1D case, in a dimensionless notation) lies in passing from the Fokker-Planck equation

$$\partial_t \rho = \Delta \rho - \nabla (b \cdot \rho), \tag{4}$$

for the probability density function $\rho(x,t)$, with the initial condition $\rho_0(x) = \rho(x,0)$ and suitable boundary data, where the existence of the stationary (equilibrium) pdf $\rho(x,t) \to \rho_*(x)$ is presumed to be granted in the large time asymptotic, to the Schrödinger-type equation i.e. the semigroup $\exp(-Ht)$:

$$\partial_t \Psi = -H\Psi = \Delta\Psi - \mathcal{V}\Psi\,, \tag{5}$$

for a real-valued function $\Psi(x,t)$. We tacitly presume the potential to be confining so that the positive definite ground state $\psi(x) \doteq \rho_*^{1/2}(x)$ exists and corresponds to the 0 eigenvalue of H. This can be always achieved by subtracting the lowest non-zero eigenvalue of H, if actually in existence, from the potential.

The auxiliary potential $\mathcal{V}$, up to an additive constant, takes the form (actually obeys the compatibility condition, given $\rho_*(x)$)

$$\mathcal{V}(x) = \rho_*^{-1/2} \Delta \rho_*^{1/2}. \tag{6}$$

The transformation between (4) and (5) is executed by means of a substitution (remember that $\rho(x.t)$, as a probability density function, integrates to 1)

$$\rho(x,t) = \Psi(x,t)\rho_*^{1/2}(x). \tag{7}$$

Another expression for the Schrödinger potential reads $\mathcal{V} = b^2/2 + \nabla b$, where $b = \nabla \ln \rho_*$, thus completing the mapping.

Ground state of H does matter !

Erratum: (1/2) factor is missing

Fokker-Planck dynamics in the interval. (Feller picture)

Under the very same infinite well conditions, after taking account of (5)-(7), another random process (devoid of any killing notion) is defined by means of the regular transition probability density (here a multiplicative Doob's transformation is involved, [13]; x and y belong to an open interval D)

$$p(t,x,y) = k(t,x,y)\frac{\rho_*^{1/2}(x)}{\rho_*^{1/2}(y)} \tag{9}$$

so that a consistent propagation of the Fokker-Planck probability density function is secured: $\rho(x,t) = \int p(t,x,y)\,\rho_0(y)\,dy$ entirely within the interval $D \subset R$. More details on these and related issues can be found in [13, 17] see also [5].

The Fokker-Planck equation (4), with a stationary solution $\rho_*(x)$, can be rewritten in the form of the general transport equation

$$\partial_t \rho = \left[\rho_*^{1/2}\Delta\left(\rho_*^{-1/2}\,\cdot\right) - \rho_*^{-1/2}\left(\Delta\rho_*^{1/2}\right)\right]\rho. \tag{10}$$

with the (motion in the interval) boundary data being implicit.

Ground state of H matters !

Remark 1: This equation often happens to be explicitly written in terms of $\rho_*(x) = \exp[-\Phi(x)]$ where Φ plays the role of the Boltzmann-Gibbs potential. In Ref. [15] we have introduced a thermal redefinition of the equilibrium pdf (self-explanatory notation): $\rho_*(x) = (1/Z)\exp[-V(x)/k_B T]$, see also [15].

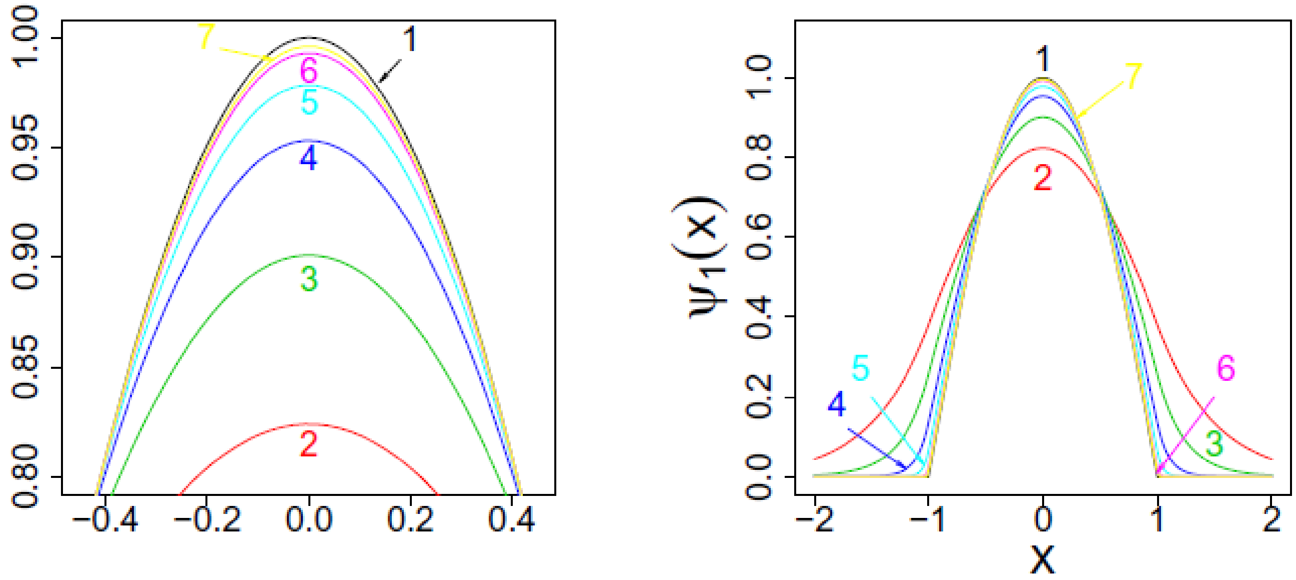


FIG. 1: Ground states $\rho_*^{1/2}$ for a sequence of deepening finite wells. Numbers refer to: 1 - $\cos(\pi x/2)$, while 2,3,4,5,6,7 enumerate well depths $V_0 = 5, 20, 100, 500, 5000, 50000$ respectively. Left panel shows an enlargement of the vicinity of maxima.

Finite well spectral problem vs infinite well, ground state and the inferred pdf

The ground state eigenvalues $E = E_1$ for various well depths have been obtained numerically and we reproduce them up to four decimal digits:

$$
\begin{array}{ll}
V_0 = 5, & E_1 = 1.1475, \\
V_0 = 20, & E_1 = 1.6395, \\
V_0 = 500, & E_1 = 2.2605, \\
V_0 = 1000, & E_1 = 2.3184, \\
V_0 = 5000, & E_1 = 2.3989, \\
V_0 = 50000, & E_1 = 2.4296, \\
V_0 \sim \infty, & E_1 = \pi^2/4 \sim 2.4674.
\end{array} \tag{14}
$$

Visual options: F -P pdf dynamics

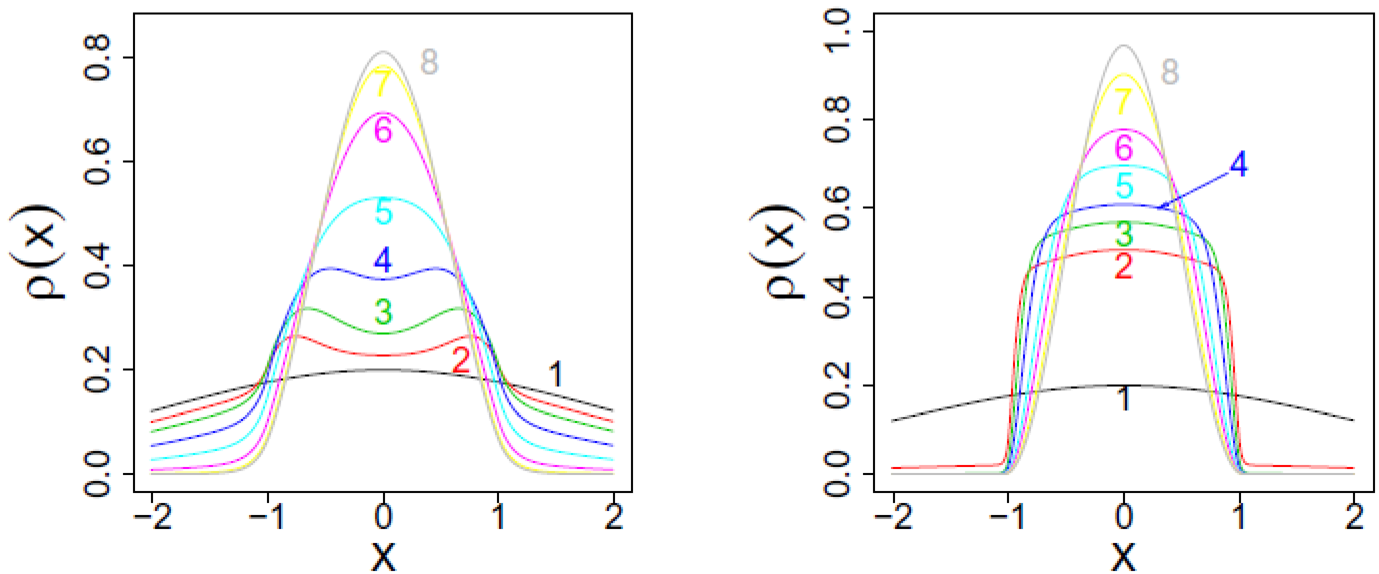


FIG. 2: Fokker-Planck dynamics of $\rho(x,t)$ in a finite well environment. It is started from the gaussian with cutoffs mentioned in the text. Numbers refer to: 1 - the gaussian (initial data), $2,3,4,5,6,7$, depict the $\rho(x,t)$ evolution at selected respective time instants (number of algorithm iteration steps). Left panel: for $V_0 = 20$ we have depicted time instants $4000, 8000, 15000, 25000, 40000, 60000$, and the vicinity of an asymptotic (8) for 120000. Right panel: for $V_0 = 1000$ the evolution proceeds somewhat faster and we have respectively $300, 600, 1200, 3000, 5000, 10000$ while 8 refers to 100000. The time increment equals $\Delta t = 10^{-5}$

Lévy stable case

Goal

Similar results for (more) general Lévy processes

- Non-locality is an issue!
- Roughly:

$$\mathcal{A}_D f = -\mu f \iff \begin{cases} \mathcal{A}f(x) = -\mu f(x) & \text{for } x \in D \\ f(x) = 0 & \text{for } x \notin D \end{cases}$$

Symmetric α-stable processes, $\Psi(\xi) = |\xi|^\alpha$

- There are $0 < \mu_1 < \mu_2 \leq \mu_3 \to \infty$ and f_n such that
$$\mathcal{A}_{(-1,1)} f_n = -\mu_n f_n$$
- $$\mu_n \sim \left(\frac{n\pi}{2}\right)^\alpha$$ (Blumenthal, Getoor)
- $$\frac{1}{2}\left(\frac{n\pi}{2}\right)^\alpha \leq \mu_n \leq \left(\frac{n\pi}{2}\right)^\alpha$$ (Chen, Song)

Theorem

$$\mu_n = \left(\frac{n\pi}{2}\right)^\alpha - \frac{(2-\alpha)\pi}{8} + O\left(\frac{1}{n}\right)$$

Set α =1, compare this with the Brownian result $\left(\frac{n\pi}{2}\right)^2$. The **Cauchy generator** $|\nabla| = (-\Delta)^{1/2}$ is not quite a square root of the negative Laplacian. What about eigenfunctions ? How does $|\nabla| = (-\Delta)^{1/2}$ act on its D =[-1,1] restricted domain ?

Finite Cauchy well: semigroup vs Feller dynamics (Robin boundary conditions ?)

> Infinite well (or infinite barrier) models make sense if they are capable of giving approximate answers to questions concerning finite wells. It is important that the validity of the approximation be controlled, which requires the notion of continuity when passing from the finite well to the infinite

We consider the Cauchy-Schrödinger semigroup dynamics $\exp(-Ht)$ where $H = T + V - E_1$ and $-T$ stands for the Cauchy generator, e.g. $T = |\nabla| = (-\Delta)^{1/2}$, while V denotes the finite well potential defined in Section II.D and E_1

is the bottom (ground state) eigenvalue of H. Here

$$T\,\psi(x) = (-\Delta)^{1/2}\,\psi(x) = \frac{1}{\pi}\int \frac{\psi(x) - \psi(x+z)}{z^2} dz,$$

The semigroup evolution gives rise to the transport equation for $\rho(x,t) = \Psi(x,t)\rho_*^{1/2}(x)$, which is a straightforward generalization of Eq. (10) mentioned in Remark 2, see for more details [12, 15, 21]:

$$\partial_t \rho = -\left[\rho_*^{1/2} T\left(\rho_*^{-1/2}\cdot\right) - \rho_*^{-1/2}\left(T\rho_*^{1/2}\right)\right]\rho. \tag{16}$$

where $\rho_*^{1/2}$ is the $L^2(R)$ normalized ground state of $H = T + V - E_1$, associated with the eigenvalue 0.

Cauchy driver in the well: pdf dynamics towards a stationary one

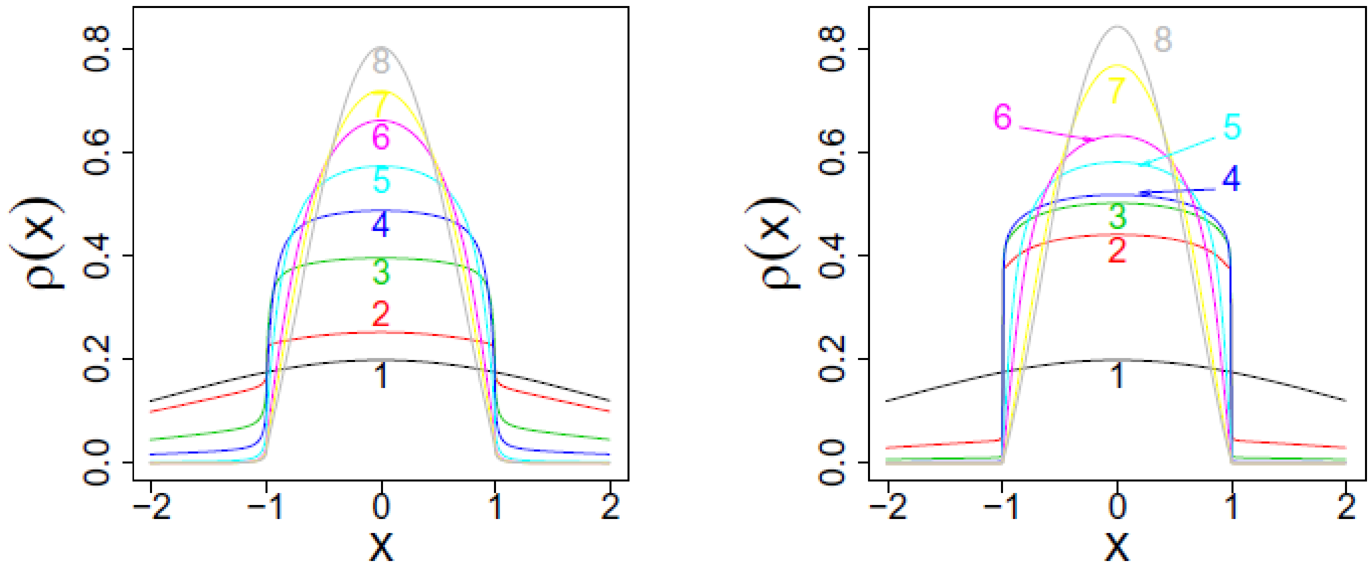


FIG. 4: Cauchy evolution of $\rho(x,t)$ in the finite well environment. Left panel $V_0 = 20$: numbers refer to: 1 - initial gaussian pdf, $2,3,4,5,6,7$, algorithmic time instants after $10,50,100,200,400,600$ steps, 8 - a close vicinity of an asymptotic pdf is approached after 2500 steps. Right panel $V_0 = 500$: 1 - initial gaussian pdf, $2,3,4,5,6,7$, refer yo $2,4,6,100,200,600$ algorithm steps respectively, 8 - a vicinity of an asymptotic pdf after 2000 steps. Time increment $\Delta t = 10^{-3}$ is 100 times larger than that adopted for Brownian simulations.

Cauchy driver: **finite well vs infinite well**

Cauchy **finite** **well** eigenvalue problem:
trigonometric connections ? Not quite …

$$V(x) = \begin{cases} 0, & |x| < 1; \\ V_0, & |x| \geqslant 1. \end{cases}$$

How distant/close are we from/ to the infinite well spectrum and eigenfunctions (e.g. ground state) , while going from the finite well depth 5 up to 500, or 5000 ?

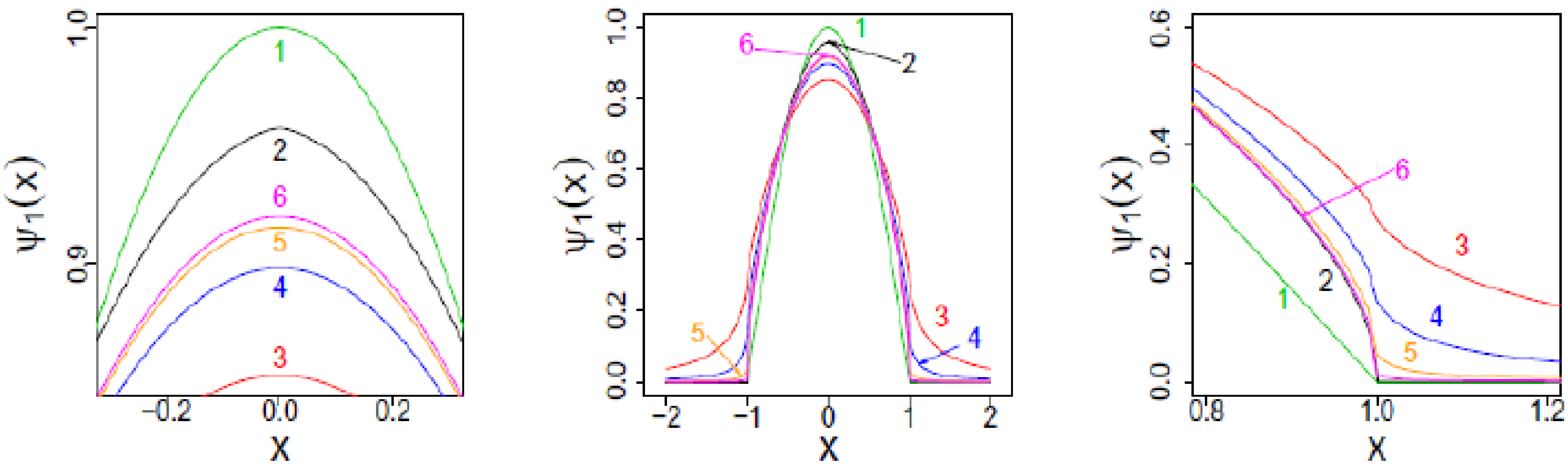


FIG. 7: Ground state solution of the Cauchy well. Numbers refer to: 1 - $\cos(\pi x/2)$, 2 - an approximate solution, Eq. (13) in [16], 3,4,5,6 refer to the well depths, respectively 5, 20, 100, 500. Convergence symptoms (towards an infinite well solution) are visually identifiable. Left panel reproduces an enlarged resolution around the maximum of the ground state. The right panel does the same job in the vicinity of the right boundary +1 of the well (curves deformation comes from scales used to increase a resolution).

Technical info: **an approximate formula for all infinite Cauchy well eigenfunctions** , according to Kwasnicki (J. Funct. Anal. 2012), Ref. (16) mentioned in previous figures)

For completeness of arguments, let us give an explicit expression for approximate eigenfuctions associated with the infinite Cauchy well. Namely, we have (with minor adjustments of the original notation of Ref. [11])

$$\psi_n(x) = q(-x)F_n(1+x) - (-1)^n q(x)F_n(1-x)\,, \qquad x \in R\,, \tag{19}$$

where $E_n = \frac{n\pi}{2} - \frac{\pi}{8}$ and $q(x)$ is an auxiliary function

$$q(x) = \begin{cases} 0 & \text{for } x \in \left(-\infty, -\frac{1}{3}\right)\,, \\ \frac{9}{2}\left(x+\frac{1}{3}\right)^2 & \text{for } x \in \left(-\frac{1}{3}, 0\right)\,, \\ 1-\frac{9}{2}\left(x-\frac{1}{3}\right)^2 & \text{for } x \in \left(0, \frac{1}{3}\right)\,, \\ 1 & \text{for } x \in \left(\frac{1}{3}, \infty\right)\,. \end{cases} \tag{20}$$

The function $F_n(x)$ is defined as follows: $F_n(x) = \sin\left(E_n x + \frac{\pi}{8}\right) - G(E_n x)$, where $G(x)$ is the Laplace transform $G(x) = \int_0^\infty e^{-xs}\gamma(s)ds$ of a positive definite function $\gamma(s)$

$$\gamma(s) = \frac{1}{\pi\sqrt{2}}\frac{s}{1+s^2}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{1}{1+r^2}\log(1+rs)dr\right)\,. \tag{21}$$

Infinite Cauchy well: ground state function problem.

Shape issue !

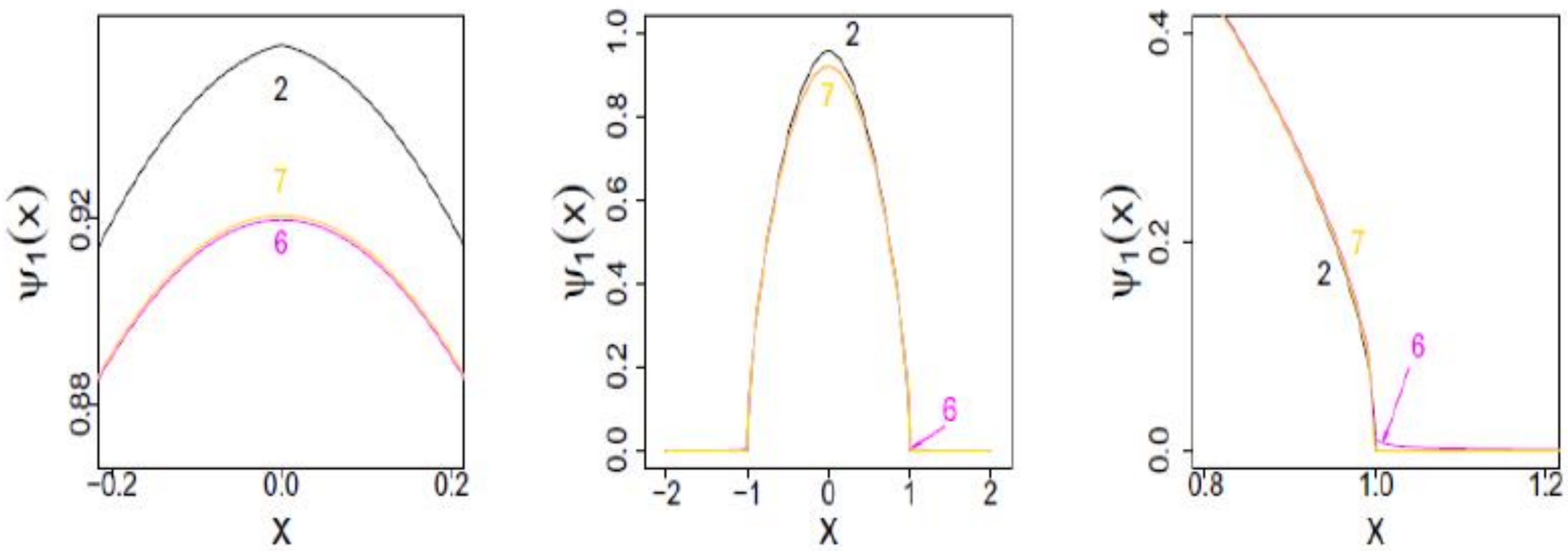


FIG. 8: Convergence towards ψ_1: 2 - an approximate ground state, Eq. (13) in [16]; Our algorithm appears to be more reliable, since 6 and 7 refer to wells whose depths are respectively 500 and 5000. Left panel shows an enlarged vicinity of the maxima. Right panel shows enlarged plots in the vicinity of +1.

Analytic guess – insightful ground state approximation

Since it is the ground state that matters in our discussion of the inferred pdf $\rho(x,t)$ dynamics, let us introduce another analytic approximation of the "true" ground state in the Cauchy case. Namely, while skipping a number of detailed hints that motivate our choice, we propose the following function as the pertinent approximation

$$\psi(x) = C\sqrt{(1-x^2)\cos(\alpha x)}, \tag{22}$$

where

$$\alpha = \frac{1443}{4096}\pi = (\frac{\pi}{2} - \frac{\pi}{8}) - \frac{\pi}{64} - \frac{\pi}{256} - \frac{\pi}{512} - \frac{\pi}{1024} - \frac{\pi}{4096}, \tag{23}$$

and $C = 0.921749$ is a normalization constant. We note that the boundary behavior of our ψ conforms with that predicted by means of scaling arguments in [2], e.g. drops down to 0 as $(1-|x|)^{1/2}$. Clearly, ψ becomes close to the cosine once away from the boundaries of $[-1.1]$. The function is concave and conforms with earlier mathematical results on the the ground state shape for stable generators in the interval, [25, 26].

Note: Shape approximation accuracy is not strictly correlated with that for the corresponding (approximate) eigenvalue. Here , we have deduced E~1.5550 while something like E~1.5777 is expected , by independent reliable reasonings .

Note that the curve
5 is out of the frame !

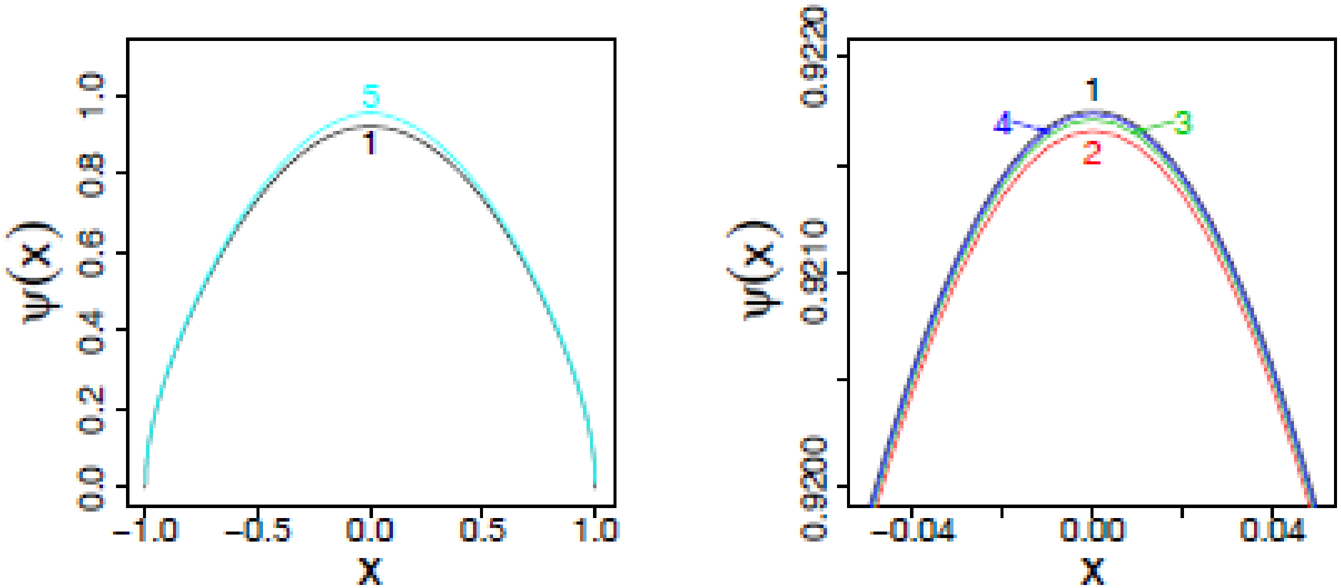


FIG. 1. Approximate ground states for Cauchy wells. Numbers refer to: 1- infinite well $\psi(x)$ of Eq.(11); 2, 3, 4 - finite wells with depths $V_0 = 5000, 10000, 20000$ respectively, [15]; 5 - infinite well proposal of [8]. In the right panel, the curve 5 is out of the frame.

Curve 5 refers to Kwasnicki approximation, curve 1 to ours

Problem: the trial function appears to be in the domain of H, **but are we really close to the true eigenfunction ?**

$$|\Delta|_D^{1/2} f = E\, f \text{ where } E \in R^+ \text{ is an eigenvalue and } f \in L^2(D)$$

Here $|\Delta|_D^{1/2}$ is the restriction of $|\Delta|^{1/2}$ to D and D=(-1,1)

$$|\Delta|^{1/2} f(x) = \frac{1}{\pi}\int_R \frac{f(x)-f(x+z)}{z^2}dz = \frac{1}{\pi}\int_R \frac{f(x)-f(z)}{|z-x|^2}dz, \quad x \in R,$$

$|\Delta|_D^{1/2} \longrightarrow A_D$

Let us tentatively consider the action of $|\Delta|^{1/2}$ on $C_0^\infty(R)$ functions $\psi(x)$, supported in $D=(-1,1)$. for all $x \in (-1,1)$ we have:

$$A_D\psi(x) = \frac{2}{\pi}\frac{\psi(x)}{1-x^2} + \frac{1}{\pi}\int_{-1-x}^{1-x}\frac{\psi(x)-\psi(x+y)}{y^2}dy.$$

Let us change the integration variable $y = t - x$ in Eq. (6). We have:

$$A_D\psi(x) = \frac{2}{\pi}\frac{\psi(x)}{1-x^2} + \frac{1}{\pi}\int_{-1}^{1}\frac{\psi(x)-\psi(t)}{(t-x)^2}dt$$

Regional fractional Laplacian

Numerically assisted **check of deviations** from the true eigenvalue formula

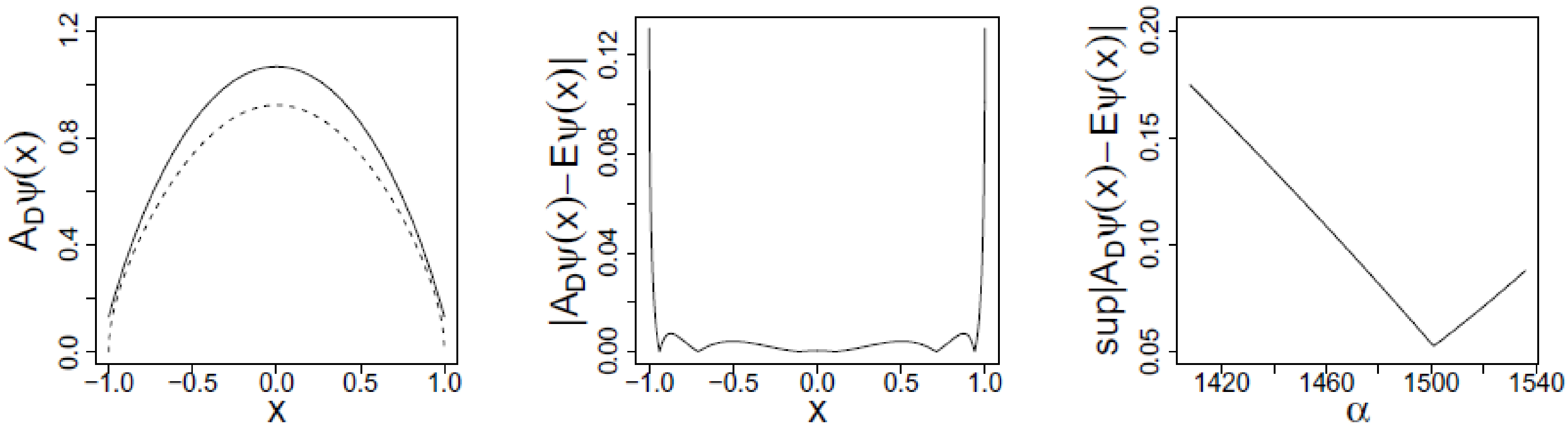


FIG. 2: Left panel: a comparison of $\psi(x) = C\sqrt{(1-x^2)cos(\alpha x)}$ (dotted line) and $A_D\psi$ (solid line). Middle panel: $|A_D\psi(x) - E\psi(x)|$ with $E = 1.156$. Right panel: supremum of $|A_D\,\psi - E\,\psi(x)|(\alpha)$ for $E = 1.156$. The α-axis is scaled in units $\pi/4096$.

POLYNOMIAL EXPANSIONS OF EIGENFUNCTIONS IN THE INFINITE CAUCHY WELL: PUSHING AHEAD APPROXIMATION FINESSE.

Then ground state function is even, hence we can expect its power series expansion in the form:

$$\psi(x) = C\sqrt{1-x^2}\sum_{n=0}^{\infty} \alpha_{2n} x^{2n}, \qquad \alpha_0 = 1.$$

where our major task is to deduce the expansion coefficients α_{2n}.

By definition we know that any solution $\psi(x)$ is defined in the domain $\bar{D} = [-1, 1]$ and obeys $\psi(\pm 1) = 0$. We extend this restriction to $A_D\psi(x)$ and demand

$$\lim_{x\to\pm 1} A_D\psi(x) = 0.$$

consider an approximation of $\psi(x)$ by series terminating at the polynomial of degree $2n$,

indicated by $w_{2n}(x)$

Effectively, we pass to:

$$\psi = C\sqrt{1-x^2}\, w_{2n}(x), \text{ with } 2n = 2, 4, 6, 10, 20, 30, 50, 70, 100, 150, 200, 500$$

Note: polynomials *w* of degree *2n* **are not** truncated „square root of the cosine" series !

Properties of approximating polynomials

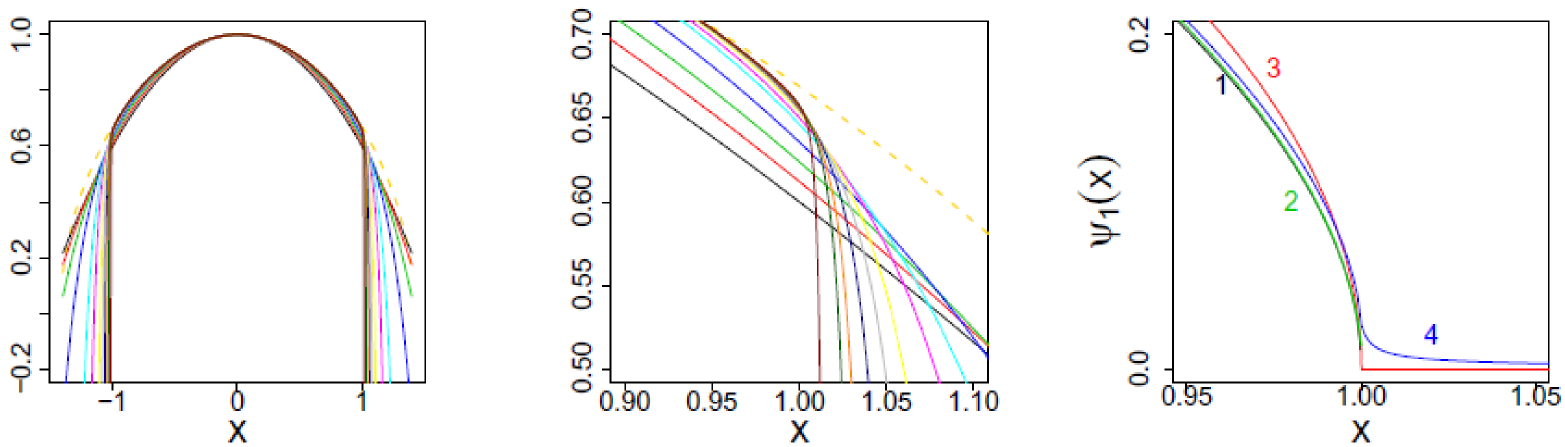


FIG. 6: Left panel: a comparative display of polynomials $w_{2n}(x)$ of degrees $2, 4, 6, 10, 20, 30, 50, 70, 100, 150, 200, 500$ and t curve $\sqrt{\cos(1443\pi x/4096)}$ (gold) which has been a building block in the formula (11). Middle panel provides an enlargeme in the vicinity of the right boundary. Right panel depicts various approximations of the ground state function at the rig boundary $x = 1$: 1 - curve $Cw_{500}(x)\sqrt{1-x^2}$, 2 - curve of [8], 3 - $\psi_1(x) \sim (1-|x|)^{1/2}$ of [2], 4 - $V_0 = 500$ finite well grou state of [15].

Note a clearly visible **deviation** from the „square root of the cosine”

Consecutive **polynomial** approximations of the ground state (we have analogous data for lowest excited states)

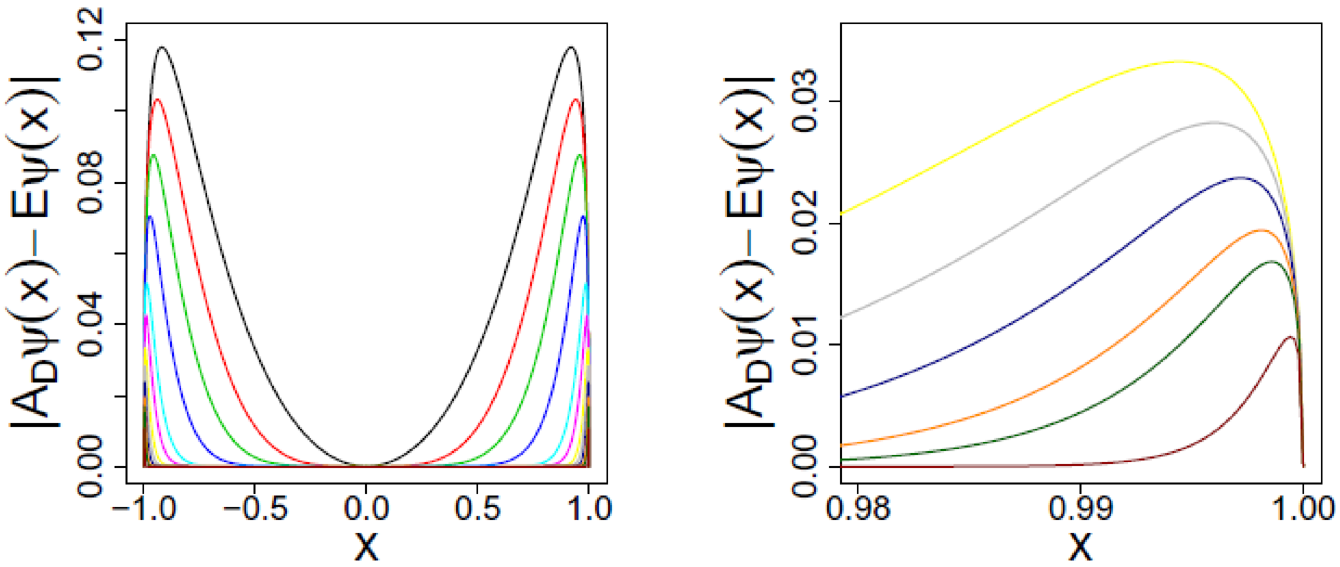
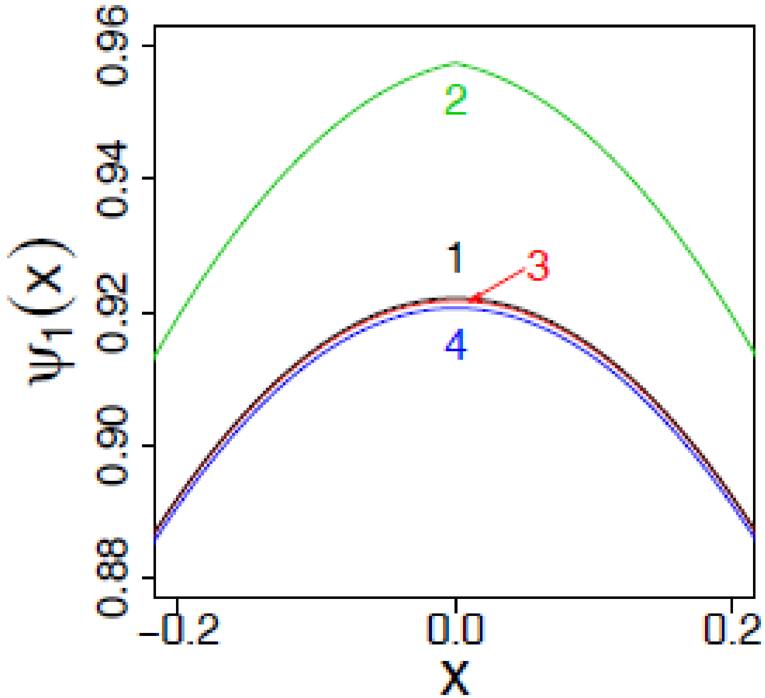


FIG. 7: Left panel: $|A_D\psi(x) - E\psi(x)|$ where $\psi = C\sqrt{1-x^2}w_{2n}(x)$, with $2n = 2, 4, 6, 10, 20, 30, 50, 70, 100, 150, 200, 500$. Right panel: polynomial degrees $2n = 50, 70, 100, 150, 200, 500$, $\psi(x)$ in the vicinity of the right boundary $x = 1$.

$$|A_D\psi(x) - E\psi(x)| \text{ where } \psi = C\sqrt{1-x^2}w_{2n}(x)$$



Various approximation outcomes: 1. Polynomial approximation of order 200 (instead of the square root of the cosine), 2. Kwasnicki curve, 3. Curve with a square root of the cosine, 4. Finite well of depth 500.

Technical note: the eigenvalue in the polynomial approximation of order 100 stopnia reads 1.1578371196122386, for order 150 we get 1.157802129761642 8 (difference of the two equals 0.000035), for order 200 we get **1.157789808316929 6** (the difference between cases 200and 150 equals 0,0000123).

Kwasnicki (2012), by means of an independent method : 1.1577738836

Cauchy operator in the interval: Technical subtleties (e.g. hypersingular integrals)

$$|\Delta|^{1/2}\psi = -\frac{1}{\pi}\left[-\psi(x)\left(\int_{-\infty}^{-1-x}\frac{dy}{y^2}+\int_{1-x}^{\infty}\frac{dy}{y^2}\right)+\int_{-1-x}^{1-x}\frac{\psi(x+y)-\psi(x)}{y^2}dy\right] =$$
$$= \frac{2}{\pi}\frac{\psi(x)}{1-x^2} - \frac{1}{\pi}\int_{-1-x}^{1-x}\frac{\psi(x+y)-\psi(x)}{y^2}dy,$$

Given $x \in (-1,1)$, let us make a substitution $x+y=t$ in (4), presuming that now the Cauchy principal value needs to be evaluated relative to x. We obtain (note the principal value ($p.v.$) symbol, introduced in the self-explanatory notation)

$$|\Delta|^{1/2}\psi = \frac{2}{\pi}\frac{\psi(x)}{1-x^2} + \frac{1}{\pi}\int_{-1}^{1}\frac{\psi(x)-\psi(t)}{(t-x)^2}dt = \frac{2}{\pi}\frac{\psi(x)}{1-x^2} + \frac{1}{\pi}(p.v.)\left[-\frac{\psi(x)}{t-x}\Big|_{-1}^{1} - \int_{-1}^{1}\frac{\psi(t)dt}{(t-x)^2}\right] =$$
$$= \frac{1}{\pi}\lim_{\epsilon\to 0}\left[\frac{2\psi(x)}{\epsilon} - \int_{-1}^{x-\epsilon}\frac{\psi(t)dt}{(t-x)^2} - \int_{x+\epsilon}^{1}\frac{\psi(t)dt}{(t-x)^2}\right] \equiv -\frac{1}{\pi}(\mathcal{H})\int_{-1}^{1}\frac{\psi(t)dt}{(t-x)^2}, \quad (5)$$

where $(\mathcal{H})$ refers to the Hadamard regularization of hypersingular integrals (Hadamard finite part, extensively employed in the engineering literature, [9]-[18]). We point out that the troublesome term $\frac{2}{\pi}\frac{\psi(x)}{1-x^2}$ has been cancelled away by its negative coming from the evaluation of $(p.v.)[...]$ in the above.

$$|\Delta|_D^{1/2}\psi(x) = -\frac{1}{\pi}(\mathcal{H})\int_{-1}^{1}\frac{\psi(t)dt}{(t-x)^2} = -\frac{1}{\pi}\frac{d}{dx}(p.v.)\int_{-1}^{1}\frac{\psi(t)dt}{t-x} = -\frac{1}{\pi}(p.v.)\int_{-1}^{1}\frac{\psi'(t)dt}{t-x}$$

The eigenvalue problem reads

$$E\,\psi(x) + \frac{1}{\pi}(p.v.)\int_{-1}^{1}\frac{\psi'(t)dt}{t-x} = 0$$

$\cos(\pi x/2)$ is not an eigenfunction of $|\Delta|_D^{1/2}$

$$|\Delta|_D^{1/2}\cos\frac{\pi x}{2} = -\frac{1}{\pi}(\mathcal{H})\int_{-1}^{1}\frac{\cos\frac{\pi t}{2}dt}{(t-x)^2} = \frac{1}{2}\cos\frac{\pi x}{2}\left[\mathrm{Si}\frac{\pi(1+x)}{2} + \mathrm{Si}\frac{\pi(1-x)}{2}\right] + \frac{1}{2}\sin\frac{\pi x}{2}\left[\mathrm{Ci}\frac{\pi(1-x)}{2} - \mathrm{Ci}\frac{\pi(1+x)}{2}\right]$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}dt = \frac{1}{\pi} - \int_x^\infty \frac{\sin t}{t}dt \qquad \mathrm{Ci}(x) = -\int_x^\infty \frac{\cos t}{t}dt = C + \ln x + \int_0^x \frac{\cos t - 1}{t}dt$$

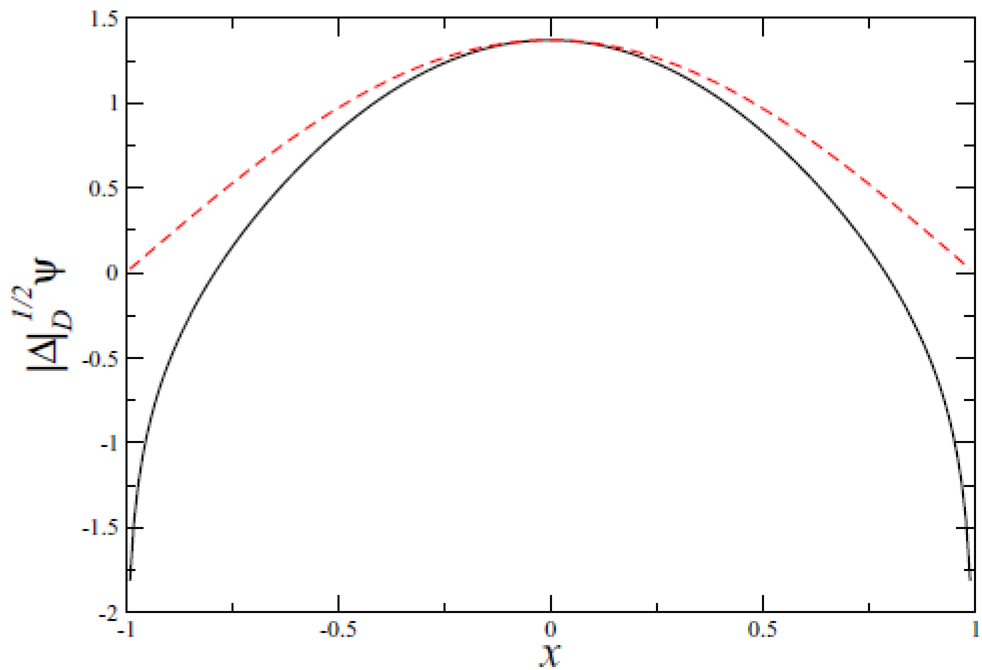


FIG. 1. The result (9) for $|\Delta|_D^{1/2}\cos(\pi x/2)$ (full line) clearly is not $E\cos(\pi x/2)$, $E > 0$ (dashed line). For demonstration purposes we have chosen $E = 1.307$ which is equal to the maximum of the function $|\Delta|_D^{1/2}\cos\frac{\pi x}{2}$

$\sin(\pi x)$ is not an (excited) eigenfunction of $|\Delta|_D^{1/2}$

$$|\Delta|_D^{1/2}\sin(\pi x) = -\frac{1}{\pi}(\mathcal{H})\int_{-1}^{1}\frac{\sin \pi t}{(t-x)^2}dt = \sin(\pi x)\Big(\mathrm{Si}[\pi(1-x)]+\mathrm{Si}[\pi(1+x)]\Big)-\cos(\pi x)\Big(\mathrm{Ci}[\pi(1-x)]-\mathrm{Ci}[\pi(1+x)]\Big)$$

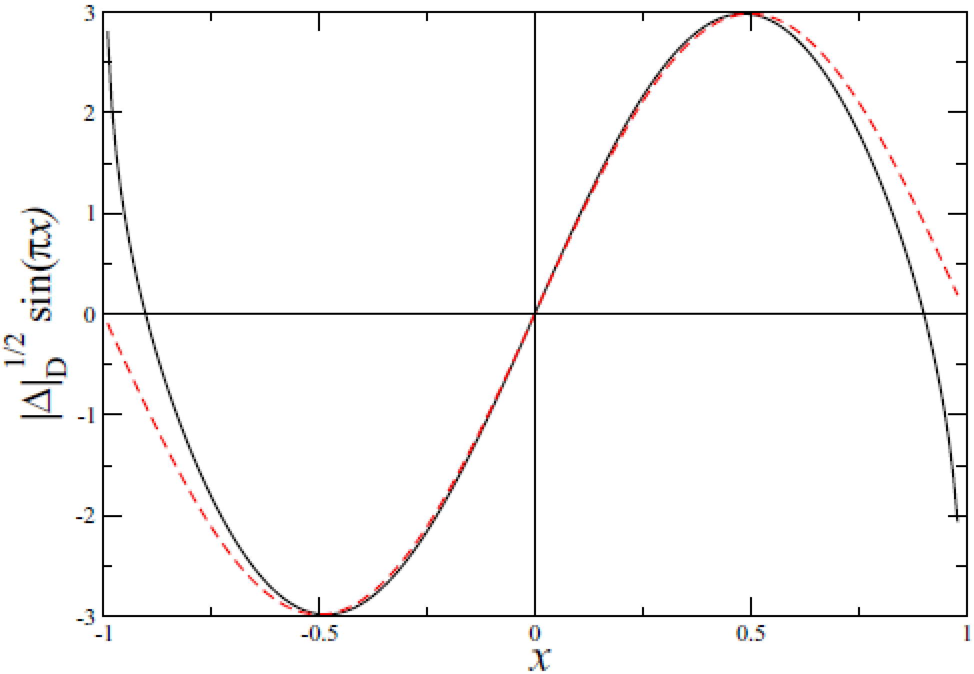


FIG. 2. The result (9) for $|\Delta|_D^{1/2}\sin(\pi x)$ (full line) is not $E'\sin(\pi x)$, $E' > 0$ (dashed line). For demonstration purposes we have chosen $E' = 2.9838$ which is equal to the maximum of $|\Delta|_D^{1/2}\sin(\pi x)$.

(Non-exhaustive) **summary of points** that need non-amateur **math**.

- **Path-wise** description of a jump-type process with killing; how do they set down at the stationary (ground) state ?

- A complementary **path-wise** desrcription of the process which equilibrates to the „ground state „ pdf (**trapping** in a finite well !)

- Maximal number of bound states in **finite wel**ls, from 1D to 3D

- More refined (as much analytic as possible) **analysis of shapes** of eigenfunctions. More accurate **eigenvalue estimates**.

- More refined analysis of **nonlocality impact** upon the computation of eigenvalues

- Spectral problems **in 3D** (partial results only for Cauchy and quasi oscillators)

Metaphysical question: Our departure point have been the jump-type processes, hence it is justified to ask what is actually jumping here . There is no „obvious" particle interpretation so much prefreered by physicists.

„Rough" conceptual guide: 1D Cauchy semigroup

J. Math. Phys. **40**, 1057, (1999)

$$\partial_t \theta_* = -|\nabla| \theta_* - V\theta_* , \qquad \partial_t \theta = |\nabla| \theta + V\theta, \tag{21}$$

where V is a measurable function such that:

(a) for all $x \in R$, $V(x) \geqslant 0$,
(b) for each compact set $K \subset R$ there exists C_K such that for all $x \in K$, V is locally bounded $V(x) \leqslant C_K$.

Lemma 5: If $1 \leqslant r \leqslant p \leqslant \infty$ and $t > 0$, then the operators T_t^V defined by

$$(T_t^V f)(x) = E_x^C \left\{ f(X_t^C) \exp\left[-\int_0^t V(X_s^C) ds \right] \right\}$$

are bounded from $L^r(R)$ into $L^p(R)$. Moreover, for each $r \in [1, \infty]$ and $f \in L^r(R)$, $T_t^V f$ is a bounded and continuous function.

Lemma 7: For any $p \in [1, \infty]$ and $f \in L^p(R)$ there holds

$$(T_t^V f)(x) = \int_R k_t^V(x,y) f(y) dy, \qquad \text{where } k_t^V(x,y) \geqslant 0 \text{ almost everywhere}$$

Lemma 8: $k_t^V(x,y)$ is jointly continuous in (x,y).

Lemma 9: $k_t^V(x,y)$ is strictly positive.

let $\rho_0(x)$ and $\rho_T(x)$ be strictly

positive densities. Then, the Markov process X_t^V characterized by the transition probability density:

$$p^V(y,s,x,t) = k_{t-s}^V(x,y)\,\frac{\theta(x,t)}{\theta(y,s)} \tag{23}$$

and the density of distributions

$$\rho(x,t) = \theta_*(x,t)\,\theta(x,t),$$

where

$$\theta_*(x,t) = \int_R k_t^V(x,y) f(y)\,dy, \qquad \theta_*(y,t) = \int_R k_{T-t}^V(x,y) g(x)\,dx$$

is precisely that interpolating Markov process to which Theorem 1 extends its validity, when the perturbed semigroup kernel replaces the Cauchy kernel.

Clearly, for all $0 \leqslant s \leqslant t \leqslant T$ we have

$$\theta_*(x,t) = \int_R k_{t-s}^V(x,y)\,\theta_*(y,s)\,dy, \qquad \theta(y,s) = \int_R k_{t-s}^V(x,y)\,\theta(x,t)\,dx \tag{24}$$

Association: set $\theta_* = \Psi$, $\theta = \rho_*^{1/2}$, so getting $\rho(x,t) = (\theta\theta_*)(x,t) = \Psi(x,t)\rho_*^{1/2}(x)$ and $\partial_t \Psi = -\hat{H}\Psi$ with $\hat{H} = |\nabla| + V$

Source papers: collaboration with M. Żaba and V. Stephanovich; 2010-2014

Physica **A 389**, 4419-4435, (2010), Levy flights in inhomogeneous environments, with V. S. (spectral solution for 1D Cauchy oscillator)

J. Math. Phys. **54**, 072103, (2013), Levy flights and nonlocal quantum dynamics, with V. S. (general framework, lots of Fourier-related discussion Cauchy wave-packet dynamics)

J. Math. Phys. **55**, 092103, (2014), Solving fractional Schrödinger-type spectral problems: Cauchy oscillator and Cauchy well, with. M. Z.

arXiv:1405.4724, Nonlocally-induced (quasirelativistic) bound states: Harmonic confinement and the finite well, with M. Z.

arXiv:1503.07458, Nonlocally induced (fractional) bound states: Shape analysis in the infnite Cauchy well (with M. Z.)

arxiv:1505.01277, Infinite Cauchy well as the hypersingular Fredholm problem