Probabilistic whereabouts of the „quantum potential”

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1. Major appearances of $\frac{\pm\Delta\rho^{1/2}}{\rho^{1/2}}$; Brownian vs quantum dynamics

2. Imaginary-time transformation: illusion of Euclidean time

3. Comments on variational extremum principles

4. Kinetic theory lore - analogies and hints

5. Heuristics of the Brownian recoil principle

$\int_R \rho(x)\,dx = 1$, **consider** $-\ln\rho(x)$, $-\nabla\ln\rho$ and $-\Delta\ln\rho$

Shannon entropy of ρ

$$\mathcal{S}(\rho) = -\langle\ln\rho\rangle = -\int \rho(x)\ln\rho(x)dx$$

Fisher information measure of ρ

$$\mathcal{F}(\rho) \doteq \langle(\nabla\ln\rho)^2\rangle = \int \frac{(\nabla\rho)^2}{\rho}dx$$

(Isoperimetric) inequality: $\qquad \mathcal{F} \geq (2\pi e)\exp(2\mathcal{S})$

$\langle\nabla\ln x\rangle = 0$ and $Var(x) = \sigma^2 = \langle(x - \langle x\rangle)^2\rangle \Longrightarrow$ an **indeterminacy rule**

$$Var(\nabla\ln x) = \mathcal{F}(\rho) \geq 1/\sigma^2 > 0$$

Information theory note: $\qquad \rho = |\psi|^2, \psi \in L^2 \Longrightarrow$

$$(1/\sigma^2) \leq \mathcal{F} \leq 16\pi^2\tilde{\sigma}^2 \qquad \textbf{and} \qquad (4\pi/\tilde{\sigma}) \leq (1/\sqrt{2\pi e})\exp[\mathcal{S}] \leq \sigma.$$

Playing with $-\ln\rho(x)$, $-\nabla\ln\rho$ and $-\Delta\ln\rho$, continued:

$$-\Delta\ln\rho = -\frac{\Delta\rho}{\rho} + \frac{(\nabla\rho)^2}{\rho^2} \Longrightarrow -\langle\Delta\ln\rho\rangle = \langle\frac{(\nabla\rho)^2}{\rho^2}\rangle = \mathcal{F}(\rho)\,.$$

For a potential of a "Newton-type force field" we have $\langle\nabla(\frac{\Delta\rho^{1/2}}{\rho^{1/2}})\rangle = 0$ and

$$-\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = \frac{1}{2}[-\frac{\Delta\rho}{\rho} + \frac{1}{2}\frac{(\nabla\rho)^2}{\rho^2}] \Longrightarrow +\nabla(\frac{\Delta\rho^{1/2}}{\rho^{1/2}}) = \frac{1}{2\rho}\nabla(\rho\Delta\ln\rho)$$

$$-\langle\frac{\Delta\rho^{1/2}}{\rho^{1/2}}\rangle = -\frac{1}{4}\langle\Delta\ln\rho\rangle = \frac{1}{4}\mathcal{F}(\rho) \geq \frac{1}{4Var(x)} > 0\,,$$

No indication of a specific physical context. However, a number of physically interesting quantities can be easily related with so-called local conservation laws for **diffusion-type processes** and the hydrodynamical formulation of the **Schrödinger picture quantum dynamics.**

Quantum hydrodynamics

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2m}\Delta + V\right]\psi$$

$\hat{H}$ self-adjoint, $\hat{H} \geq 0$, $\rho(x,t) = |\psi|^2(x,t)$, $v = (\hbar/2mi)[(\nabla\psi/\psi) - (\nabla\psi^*/\psi^*)] \Longrightarrow$

$$\partial_t\rho = -\nabla(\rho v); \qquad \partial_t s + \frac{1}{2m}(\nabla s)^2 + (V + Q) = 0 \Longrightarrow$$

$$\partial_t v + (v\nabla v) = -\tfrac{1}{m}\nabla(V + Q)$$

where $v = \frac{1}{m}\nabla s$ **and** $Q = Q[\rho] = -\frac{\hbar^2}{2m}\frac{\Delta\rho^{1/2}}{\rho^{1/2}}$.

Set $|\psi| = \rho_*^{1/2}$. The **ground state** condition for $\hat{H}$ reads: $V = +\frac{\hbar^2}{2m}\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = -Q[\rho_*]$.

Denote $u(x,t) \doteq (\hbar/2m)\nabla\ln\rho$. Dynamics arises via $\{\rho, s\}$ **extremum** of $I(\rho, s) = \int_{t_1}^{t_2}\langle\left[\partial_t s + \frac{m}{2}(u^2 + v^2) + V\right]\rangle(t)dt$.

In terms of valid solutions $\rho(x,t)$, $s(x,t)$, we arrive at a strictly positive constant of motion: $-\langle\partial_t s\rangle = H = \langle\left[\frac{m}{2}(u^2 + v^2) + V\right]\rangle > 0$ (**finite energy condition**).

Brownian hydrodynamics

$$\exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t \implies \partial_t \Psi = \left[D\Delta - \frac{V}{2mD}\right]\Psi$$

$\hat{H}$ self-adjoint, $\hat{H} \geq 0$, $t \geq 0$. Set $\hbar \equiv 2mD$. Let $\Psi(x,t) \to \rho_*^{1/2}$ as $t \to \infty$. Define $\rho(x,t) = \Psi(x,t)\rho_*^{1/2}(x)$ with $b = D\nabla \ln \rho_*$, $u = D\nabla \ln \rho$, $v = b - u = (1/m)\nabla s$.

$$\frac{V(x)}{2mD} = +D\frac{\Delta\rho_*^{1/2}}{\rho_*^{1/2}} \doteq mD\left[\frac{b^2}{2D} + \nabla b\right] \Longrightarrow$$

$$\partial_t \rho = D\Delta\rho - \nabla(b\rho) \Longleftrightarrow \partial_t \rho = -\nabla(v\rho)$$

$$\partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0 \implies \partial_t v + (v\nabla v) = +\frac{1}{m}\nabla(V + Q)$$

Note a **compatibility condition** $V \equiv -Q[\rho_*]$. The $\{\rho, s\}$ **extremum principle** for $I(\rho, s) = \int_{t_1}^{t_2}\langle\big[\partial_t s + (m/2)(v^2 - u^2) - V\big]\rangle$ yields equations of motion. On dynamically admitted fields $\rho(t)$ and $s(x,t)$, we have $-\langle\partial_t s\rangle = H = \langle\big[\frac{m}{2}(v^2 - u^2) - V\big]\rangle \equiv 0$!

Dynamical duality - illusion of "Euclidean time"

$$it \to t \geq 0; \quad \hbar \to 2mD$$

$$\exp(-i\hat{H}t/\hbar)\psi_0 = \psi_t \longrightarrow \exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t$$

Given the spectral solution for $\hat{H} = -\Delta + V$, the integral kernel of $\exp(-t\hat{H})$ reads

$$k(y, x, t) = \sum_j \exp(-\epsilon_j t)\, \Phi_j(y)\Phi_j^*(x)$$

Remember that $\epsilon_0 = 0$ and the sum may be replaced by an integral in case of a continuous spectrum, (with complex-valued generalized eigenfunctions). **Set** $V(x) = 0$.

$$k(y, x, t) = [\exp(t\Delta)](y, x) = (2\pi)^{-1/2} \int \exp(-p^2 t)\, \exp(ip(y - x)\, dp =$$

$$(4\pi t)^{-1/2} \exp[-(y - x)^2/4t]$$

Consider $\hat{H} = (1/2)(-\Delta + x^2 - 1)$ (e.g. rescaled harmonic oscillator Hamiltonian). The integral kernel of $\exp(-t\hat{H})$ is given by the classic Mehler formula:

$$k(y,x,t) = k(x,y,t) = [\exp(-t\hat{H})(y,x) =$$

$$[\pi(1-\exp(-2t))^{-1/2}\exp[-(1/2)(x^2-y^2) - (1-\exp(-2t))^{-1}(x\exp(-t)-y)^2]$$

The normalization condition $\int k(y,x,t)\exp[(y^2-x^2)/2]\,dy = 1$ actually defines the transition probability density of the Ornstein-Uhlenbeck process

$$p(y,x,t) = k(y,x,t)\,\rho_*^{1/2}(x)/\rho_*^{1/2}(y)$$

with $\rho_*(x) = \pi^{-1/2}\exp(-x^2)$. A more familiar form of the kernel reads (note the presence of $\exp(t/2)$ factor)

$$k(y,x,t) = \frac{\exp(t/2)}{(2\pi\sinh t)^{1/2}}\exp\left[-\frac{(x^2+y^2)\cosh t - 2xy}{2\sinh t}\right]$$

Execute $t \to it$. We get a free Schrödinger propagator

$$K(y,x,t) = [\exp(it\Delta)](y,x) = (2\pi)^{-1/2} \int \exp(-ip^2 t)\ \exp(ip(y-x)\,dp =$$

$$(4\pi it)^{-1/2}\ \exp[+i(y-x)^2/4t]$$

and likewise, that of (here -1 renormalized) harmonic oscillator propagator

$$K(y,x,t) = \frac{\exp(it/2)}{(2\pi i \sin t)^{1/2}} \exp\left[+i \frac{(x^2+y^2)\cos t - 2xy}{2\sin t}\right]$$

Learn a standard **Euclidean** (field) theory lesson concerning **multi-time correlation functions**; exemplary harmonic oscillator case, $t > t' > 0$; $t \to it$:

$$E[X(t')X(t)] = \int \rho_*(x')\, x'\, p(x',t',x,t)\, x\, dz dx' = (1/2)\ \exp[-(t-t')] \Longrightarrow$$

$$W(t',t) = \langle \psi_0, \hat{q}_H(t)\hat{q}_H(t')\psi_0\rangle = (1/2)\exp[-i(t-t')]$$

Note: This appealing correspondence **breaks down** in R^3, in the presence of electromagnetic fields !

Comments on variational extremum principles

1. (Shannon) Entropy extremum principle: Given $V = V(x)$, fix a priori $\langle V \rangle = \zeta$. Extremize $\mathcal{S} = -\langle \ln \rho \rangle$ under this **constraint**: seek an extremum of

$$\mathcal{S}(\rho) + \alpha \langle V \rangle = \langle -\ln \rho + \alpha V \rangle$$

where α is a Lagrange multiplier. **Outcome:** α-family of pdfs $\rho_\alpha = A_\alpha \exp[\alpha V(x)]$ arises, provided $(A_\alpha)^{-1} = \int \exp[\alpha V(x)]\, dx$ exists; α-value comes from $\langle V \rangle_\alpha = \zeta$.

2. Fisher information extremum principle: Fix a priori $\langle V \rangle = \zeta$. Extremize the Fisher information measure $\mathcal{F}(\rho)$ under that **constraint**:

$$\mathcal{F}(\rho) + \lambda \langle V \rangle = \langle (\nabla \ln \rho)^2 + \lambda V \rangle$$

Remember that $-\langle \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \rangle = \frac{1}{4} \mathcal{F}(\rho)$. The extremizing pdf $\rho(x) \doteq \rho_*(x)$ comes out from :

$$V(x) = \frac{2}{\lambda} [\frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2}] = + \frac{4}{\lambda} \frac{\Delta \rho^{1/2}}{\rho^{1/2}}$$

Outcome: λ-family of pdfs; λ gets fixed by $\langle V \rangle_\lambda = \zeta$. Setting $\lambda = 2/mD^2$, we recover the **Brownian** framework; $\lambda = 8m/\hbar^2$ is admitted as a **special case**.

Hamilton-Jacobi route

$H = p^2/2m + V(x)$, $\{\dot{q} = p/m,\ \dot{p} = -\nabla V(q)\}$; assign $\rho_0(x)$ ($\Rightarrow \mathcal{S}(\rho)$ and $\mathcal{F}(\rho)$).

$$I_0(\rho, s) = \int_{t_1}^{t_2} \langle \left[\partial_t s + \frac{1}{2m}(\nabla s)^2 + V \right] \rangle \, dt \Longrightarrow \partial_t s + \frac{1}{2m}(\nabla s)^2 + V = 0$$

and $\partial_t \rho = -\nabla(v\rho)$. Here, $v = (1/m)\nabla s$ and we have $\partial_t v + (v\nabla v) = -\nabla V$.

Constrained Fisher information: Fix a priori $\int_{t_1}^{t_2} \mathcal{F}(\rho)(t)\, dt = \zeta$. Extremize

$$I_\gamma(\rho, s) = \int_{t_1}^{t_2} dt \, \langle \left[\partial_t s + \frac{(\nabla s)^2}{m} \pm V \right] + \gamma \frac{(\nabla \rho)^2}{\rho^2} \rangle \Longrightarrow$$

$$\partial_t \rho = -\nabla(v\rho)$$

$$\partial_t s + \frac{(\nabla s)^2}{m} \pm V + 4\gamma \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = 0$$

$\pm V$ is intended to make a distinction between confining and scattering potentials.

Outcomes (admissible case of $\gamma = 0$ is left aside):
(i) $\gamma = -mD^2/2$, eventually followed by setting $D = \hbar/2m$, leads to the D-labelled **quantum hydrodynamics** (before, we have referred to $+V$ only)

$$\partial_t s + \frac{1}{2m}(\nabla s)^2 \pm V + Q = 0$$

(ii) $\gamma = +mD^2/2$, with the potential term $-V$ only, leads to the **Brownian hydrodynamics**

$$\partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0$$

Note: $t \to it$ relationship can be secured for $+V$, where V is a confining potential.

$$\partial_t s + \frac{1}{2m}(\nabla s)^2 + (V + Q) = 0$$

c.f. $t \to it \Longrightarrow \exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t \longrightarrow \exp(-it\hat{H}/2mD)\psi_0 = \psi_t$ issue.
We demand $\hat{H}$ to have a bottom eigenvalue equal zero (to yield a contractive semigroup). For a bounded from below Hamiltonian this can be always achieved, like e.g. in case of $\hat{H} = (1/2)(-\Delta + x^2 - 1)$.

Hamilton-Jacobi route - a catalogue of "standards"

(i) $\mathcal{L}^{+} = -\rho\left[\partial_t s + (m/2)(v^2 + u^2) + V\right] \Longrightarrow \partial_t s + (1/2m)(\nabla s)^2 + (V + Q) = 0$

(ii) $\mathcal{L}^{\pm}_{cl} = -\rho\left[\partial_t s + (m/2)v^2 \pm V\right] \Longrightarrow \partial_t s + (1/2m)(\nabla s)^2 \pm V = 0$

(iii) $\mathcal{L}^{-} = -\rho\left[\partial_t s + (m/2)(v^2 - u^2) - V\right] \Longrightarrow \partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0.$

On dynamically admitted fields $\rho(t)$ and $s(x,t)$, $L(t) = \int dx\, \mathcal{L} \sim 0$, i.e. $\langle \partial_t s \rangle = -H$. The respective **Hamiltonians** obey:

(i) $H^{+} \doteq \int dx\, \rho\left[(m/2)v^2 + V + (m/2)u^2\right] > 0$, (quantum) constant of motion !

(ii) $H^{\pm}_{cl} \doteq \int dx\, \rho\left[(m/2)v^2 \pm V\right] = E, \quad E = (p^2/2m) \pm V(x)$, constant on each path

(iii) $H^{-} \doteq \int dx\, \rho\left[(m/2)v^2 - V - (m/2)u^2\right] = 0$, identically (!) in Brownian motion

We emphasize that, from the start, $V(x)$ is chosen to be **confining** (a class of continuous and bounded from below functions allows to secure $\hat{H} \geq 0$).

Kinetic theory lore: Brownian analogies and hints

Consider **free phase-space Brownian motion** in the large friction regime. $W(x,u,t)$ stands for phase-space (velocity-position) probability distribution with suitable initial data at $t=0$. Denote $w(u,t)$ and $w(x,t)$, the marginal pdfs.

We set $D = k_B T/m\beta$ and observe that actually, in the large friction regime, $w(x,t) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt)$ solves $\partial_t w = D\Delta w$.

$\langle u\rangle = \int du\, u\, W(x,u,t) \to \langle u\rangle = (x/2t)w(x,t)$

$\langle u\rangle_x = \langle u\rangle / w(x,t) = x/2t = -D(\nabla w)/w$

$\langle u^2\rangle_x = \langle u^2\rangle / w(x,t) = (D\beta - D/2t) + \langle u\rangle_x^2$

The Kramers-Fokker-Planck equation

$$\partial_t W + u\nabla_x W = \beta \nabla_u (W u) + q\Delta_u W$$

with $q = D\beta^2$, implies the local conservation laws

$$\partial_t w + \nabla(\langle u\rangle_x w) = 0$$

$$\partial_t(\langle u\rangle_x w) + \nabla_x(\langle u^2\rangle_x w) = -\beta\langle u\rangle_x w$$

Introducing the **kinetic pressure** $P_{kin}(x,t) = [\langle u^2\rangle_x - \langle u\rangle_x^2]w(x,t)$ we arrive at

$$\partial_t + \langle u\rangle_x \nabla)\langle u\rangle_x = -\beta \langle u\rangle_x - \nabla P_{kin}/w$$

In the large friction regime we have

$$-\frac{\nabla P_{kin}}{w} = +\beta\langle u\rangle_x - \frac{\nabla P_{osm}}{w}$$

where $P_{osm} = D^2 w \Delta \ln w$ we name an **osmotic pressure** in the Brownian motion.

$$\nabla P_{osm} = -w\nabla Q/m \qquad \text{with} \qquad Q = -2mD^2 \frac{\Delta w^{1/2}}{w^{1/2}}$$

Actually $-\nabla P_{osm} = (D/2t)\nabla w$. Thus, denoting $\langle u\rangle_x = v(x,t)$ we arrive at:

$$(\partial_t + v\nabla)v = -\frac{\nabla P_{osm}}{w} = +\frac{1}{m}\nabla Q$$

to be compared with the general **Brownian hydrodynamics** result

$$\partial_t v + (v\nabla v) = +\frac{1}{m}\nabla(V+Q)$$

In the past (1992) I have named all that: "derivation of the quantum potential from realistic Brownian particle motions".

Concerning the pressure terms P_{kin} and P_{osm}

In view of $-\langle \Delta \ln \rho \rangle = \mathcal{F}(\rho) > 0$, P_{osm} is predominantly **negative-definite**. To the contrary, P_{kin} is **positive definite**, hence the large friction regime is valid for times $t > (2\beta)^{-1}$. Let us introduce the **kinetic temperature**:

$$0 \leq \Theta_{kin} = m\frac{P_{kin}}{w} \sim (k_B T - \frac{mD}{2t}) < k_B T$$

whose (large time limit) asymptotic value, $k_B T$ actually is. Since $P_{osm}/w = D^2 \Delta \ln w = -D/2t$, we learn that a (predominantly !) positive-definite quantity

$$\Theta_{osm} = -m\frac{P_{osm}}{w} = -mD^2 \Delta \ln w \Longrightarrow \Theta_{kin} \sim (k_B T - \Theta_{osm})$$

gives account of the **deviation from thermal equilibrium** in terms of the local "thermal energy" (agitation) Θ_{osm}.
One more useful identity (not an independent equation) is valid. It expresses the "thermal energy" conservation law (**no** thermal currents are hereby induced):

$$(\partial_t + v\nabla)\Theta_{osm} = -2(\nabla v)\Theta_{osm} \Longrightarrow \partial_t \Theta_{osm} = -2(\nabla v)\Theta_{osm}$$

Meaning of the pressure term in Brownian hydrodynamics ($P_{osm} \doteq P$)

$$\partial_t v + (v\nabla v) = +\frac{1}{m}\nabla(V+Q) = \frac{1}{m}F - \frac{\nabla P}{w}; \qquad -\frac{\nabla P}{w} = +\frac{1}{m}\nabla Q; \;\; F \doteq -\nabla(-V)$$

In normal liquids the pressure is exerted upon any control volume (**droplet**) $\Rightarrow$ a **compression** of a droplet. In case of Brownian motion, we deal with a definite **decompression.**

Consider a reference volume (control interval, finite **droplet**) $[\alpha,\beta]$ in R^1 (or $\Lambda \subset R^1$) which at time $t > 0$ comprises a certain fraction of particles (Brownian "fluid" constituents).

The **time rate of particles loss or gain** by the volume $[\alpha,\beta]$ at time t, is equal to the flow outgoing through the boundaries

$$-\partial_t \int_\alpha^\beta \rho(x,t)dx = \rho(\beta,t)v(\beta,t) - \rho(\alpha,t)v(\alpha,t)$$

To analyze the **momentum balance**, let us slightly deform the boundaries $[\alpha,\beta]$ to compensate the mass imbalance: $[\alpha,\beta] \to [\alpha + v(\alpha,t)\triangle t, \beta + v(\beta,t)\triangle t]$. Effectively, we pass to a locally co-moving (**droplet**) frame (that is the Lagrangian picture).

(i) The **mass balance** in the moving **droplet** has been achieved:

$$lim_{\triangle t\downarrow 0}\frac{1}{\triangle t}[\int_{\alpha+v_\alpha\triangle t}^{\beta+v_\beta\triangle t}\rho(x,t+\triangle t)dx-\int_\alpha^\beta\rho(x,t)dx]=0$$

(ii) For local matter flows $(\rho v)(x,t)$, in view of $\partial_t(\rho v)=-\nabla(\rho v^2)+(1/m)\rho\nabla(V+Q)$, the **rate of change of momentum** (per unit of mass) of the droplet, reads

$$lim_{\triangle t\downarrow 0}\frac{1}{\triangle t}[\int_{\alpha+v_\alpha\triangle t}^{\beta+v_\beta\triangle t}(\rho v)(x,t+\triangle t)-\int_\alpha^\beta(\rho v)(x,t)]=\int_\alpha^\beta\rho\frac{1}{m}\nabla(V+Q)dx$$

However, $\nabla Q/m=-\frac{\nabla P}{\rho}$ and $P=D^2\rho\triangle ln\rho$. Therefore:

$$\int_\alpha^\beta\rho\frac{1}{m}\nabla(V+Q)dx=\int_\alpha^\beta\rho\nabla\Omega dx-\int_\alpha^\beta\nabla Pdx=\frac{1}{m}E[\nabla V]_\alpha^\beta+P(\alpha,t)-P(\beta,t)$$

(iii) The **time rate of change of the kinetic energy** of the droplet is:

$$lim_{\triangle t\downarrow 0}\frac{1}{\triangle t}[\int_{\alpha+v_\alpha\triangle t}^{\beta+v_\beta\triangle t}\frac{1}{2}(\rho v^2)(x,t+\triangle t)-\int_\alpha^\beta\frac{1}{2}(\rho v^2)(x,t)]=\int_\alpha^\beta\frac{1}{m}(\rho v)\nabla(V+Q)dx$$

Note that $\int_\alpha^\beta\rho v\nabla Qdx=-\int_\alpha^\beta v\nabla Pdx$ (c.f. the notion of power release $\frac{dE}{dt}=F\cdot v$)

Meaning of the pressure term in quantum hydrodynamics ($-P_{osm} \doteq P$)

$$\partial_t v + (v\nabla v) = -\frac{1}{m}\nabla(V+Q) = \frac{F}{m} - \frac{\nabla P}{\rho} \Longrightarrow$$

$$-\frac{1}{m}\nabla Q = +\frac{\nabla P_{osm}}{\rho} \doteq -\frac{\nabla P}{\rho}$$

which enforces $-P_{osm} = -D^2\rho\Delta \ln\rho \doteq P$, $D = \hbar/2m$, while $F = -\nabla V$. If compared to the Brownian hydrodynamics **all** $(V+Q)$ contributions come with an **inverted sign**. That carries over to the mass, momentum and kinetic energy rates. **Contrary** to the Brownian $P = P_{osm}$, the quantum pressure term $P = -P_{osm}$ is predominantly positive. We recall that $-\langle\Delta\ln\rho\rangle = \langle\frac{(\nabla\rho)^2}{\rho^2}\rangle = \mathcal{F}(\rho) > 0$.
We note in passing that quantum mechanically derivable heat transfer equation

$$(\partial_t + v\nabla)\Theta_{osm} = -2\frac{\nabla q}{\rho} - 2(\nabla v)\Theta_{osm}$$

with $\Theta_{osm} = -m\frac{P_{osm}}{\rho} = -mD^2\Delta\ln\rho$ and $q = -2mD^2\rho\Delta v$, reproduces the Brownian form, at least for generic free Schrödinger wave packets with $\Delta v = 0$. We get $\partial_t\Theta_{osm} = -2(\nabla v)\Theta_{osm}$ as well. There is **no** heat current.

Hamilton-Jacobi related hydrodynamics and (Bohmian) trajectory descriptions

Eulerian picture (passive control) vs **Lagrangian** picture (active control in a co-moving frame): simply give our previous droplet (co-moving control volume) an infinitesimal size. We get **droplet dynamics** along Bohm-type trajectories.

$$f(x,t) \to f(x(t+\Delta t), t+\Delta t) \sim [\partial_t f + (v\nabla)f]\Delta t; \quad \dot{x} = v = v(x,t)_{|x(t)=x}$$

$$x(t+\Delta t) \sim v\Delta t,\ v = (1/m)\nabla s \text{ and } \partial_t s = \tfrac{ds}{dt} - mv^2 \text{ imply}$$

(i) Classical hydrodynamics: (droplet) paths in the Lagrangian frame

$$\frac{d\rho}{dt} = -\rho\nabla v \longrightarrow \rho(x(t+\Delta t), t+dt) \sim \exp[-(\nabla v)\Delta t]\,\rho(x,t)$$

$$\frac{ds}{dt} = \frac{1}{2m}(\nabla s)^2 - (\pm V) \Longrightarrow m\frac{dv}{dt} = -\nabla(\pm V)$$

(ii) Brownian hydrodynamics: (droplet) paths in the Lagrangian frame

$$\frac{d\rho}{dt} = -\rho \nabla v$$

$$\frac{ds}{dt} = \frac{1}{2m}(\nabla s)^2 + (V+Q) \Longrightarrow m\frac{dv}{dt} = +\nabla(V+Q)$$

Purely random (Wiener) background: $dX(t) = b(X(t))dt + \sqrt{2D}dW(t) \Longrightarrow \partial_t \rho = D\Delta\rho - \nabla(b\rho)$; $\frac{V(x)}{2mD} = mD\left[\frac{b^2}{2D} + \nabla b\right]$

(iii) Quantum hydrodynamics: (droplet) paths in the Lagrangian frame $\Longrightarrow$ Bohmian trajectories

$$\frac{d\rho}{dt} = -\rho \nabla v$$

$$\frac{ds}{dt} = \frac{1}{2m}(\nabla s)^2 - (V+Q) \Longrightarrow m\frac{dv}{dt} = -\nabla(V+Q)$$

Back to random paths: diffusion-type processes

Consider a Markovian diffusion process on R, for times $t \in [0,T]$: $dX(t) = b(X(t),t)dt + \sqrt{2D}dW(t)$, where $W(t)$ stands for the Wiener noise and $X(t_0) = x_0$. Given $p(y,s,x,t), s \leq t$ and $\rho_0(x)$, we can infer a **statistical future** of the process:

$$\rho(x,t) = \int \rho(y,s)p(y,s,x,t)dy \Longrightarrow \partial_t \rho = D\Delta - (\nabla b\rho)$$

$$b(x,t) = \lim_{\Delta \to 0} \frac{1}{\Delta t} \int (y-x)p(x,t,y,t+\Delta t)dy = v(x,t) + (D\nabla \rho/\rho)(x,t)$$

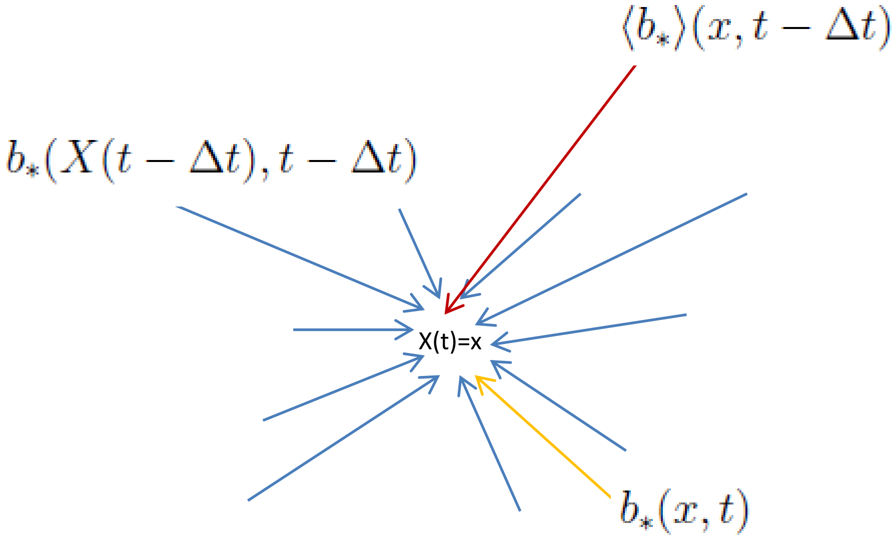
We can as well reproduce a **statistical past** of the process, by means of

$$p_*(y,s,x,t) \doteq p(y,s,x,t)\frac{\rho(y,s)}{\rho(x,t)} \Longrightarrow \rho(y,s) = \int p_*(y,s,x,t)\rho(x,t)dx$$

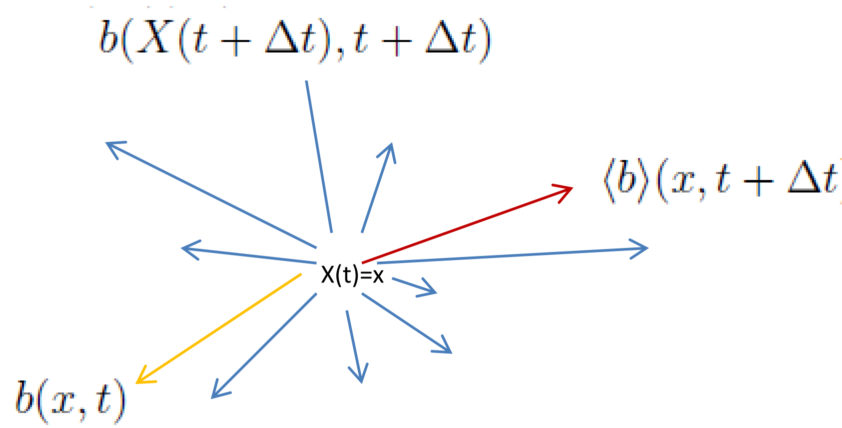
$$b_*(y,s) = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \int (y-s)p_*(x,s-\Delta s,y,s)dx = v(y,s) - (D\nabla \rho/\rho)(y,s)$$

Making notice of $v = (1/2)(b + b_*)$, we get:

$$\partial_t \rho = -\nabla(v\rho) = D\Delta\rho - (\nabla b\rho) = -D\Delta\rho - \nabla(b_*\rho)$$



Incoming random flow



Outgoing random flow

Impulsive behavior of drifts in Brownian motion

$$b_*(x,t) - \langle b_* \rangle (x, t - \Delta t) \sim \langle b \rangle (x, t + \Delta t) - b(x,t) \sim \frac{1}{m} \nabla V \, \Delta t$$

Impulsive behavior of drifts in stochastic mechanics

$$b_*(x,t) - \langle b_* \rangle (x, t - \Delta t) \sim \langle b \rangle (x, t + \Delta t) - b(x,t) \sim \frac{1}{m} \nabla (V + 2Q) \, \Delta t$$

Consider $b = DX$ and $b_* = D_*X$ as special cases of forward (**predictive**) and backward (**retrodictive**) time derivatives of functions of the random variable $X(t)$:

$$(Df)(X(t),t) = (\partial_t + b\nabla + D\Delta)f; \quad (D_*f)(X(t),t) = (\partial_t + b_*\nabla - D\Delta)f$$

Analyze acceleration formulas for diffusion-type processes.

II^{nd} Newton law in the (local) mean

(i) Brownian motion

$$(D^2X)(t) = (\partial_t + v\nabla)v - \frac{1}{m}\nabla Q = (D_*^2X)(t) = +\frac{1}{m}\nabla V$$

(ii) Nelson's stochastic mechanics

$$\frac{1}{2}[(DD_* + D_*D)X](t) = (\partial_t + v\nabla)v + \frac{1}{m}\nabla Q = -\frac{1}{m}\nabla V$$

Something is conspicuously missing: set $\nabla V = 0$, still accelerating ! $\Longrightarrow$

$$\text{(i) } \frac{dv}{dt} - \frac{1}{m}\nabla Q = 0, \qquad \text{(ii) } \frac{dv}{dt} + \frac{1}{m}\nabla Q = 0$$

$\mp\nabla Q$ contributes to the $\pm\Delta v$ velocity increment as a legitimate **force.**

Our preference for the II^{nd} Newton law is:

$$\text{(iii) } (\partial_t + v\nabla)v = \pm\frac{1}{m}\nabla(Q + V)$$

Brownian impulse (times Δt)

$$(D^2X)(t) = (D_*^2X)(t) = +\tfrac{1}{m}\nabla V \Longleftrightarrow \tfrac{1}{2}[(DD_* + D_*D)X](t) = \tfrac{1}{m}\nabla(V + 2Q)$$

Stochastic mechanics impulse (times Δt)

$$(D^2X)(t) = (D_*^2X)(t) = -\tfrac{1}{m}\nabla(V + 2Q) \Longleftrightarrow \tfrac{1}{2}[(DD_* + D_*D)X](t) = -\tfrac{1}{m}\nabla V$$

$\Downarrow$ $\quad III^{rd}$ Newton law in the mean: $\pm\dfrac{2}{m}\nabla(V+Q)$ $\quad\Uparrow$

Impulse-momentum change equations

Brownian impulse in a co-moving frame (given ρ and v)

$$\Delta\rho = -[(\nabla v)\,\Delta t]\,\rho \qquad\qquad m\Delta v = +\nabla(V+Q)\,\Delta t$$

Anti-Brownian impulse in a co-moving frame (given ρ and v)

$$\Delta\rho = -[(\nabla v)\,\Delta t]\,\rho \qquad\qquad m\Delta v = -\nabla(V+Q)\,\Delta t$$

Introducing the **„Brownian recoil principle"**

C.f. physics of firearms (note a recoiless gun) ; I am sorry for military associations

Brownian recoil principle

Consider $\Delta t \ll 1$. Within $[t, t+\Delta t]$, let the **action-reaction coupling** between "vacuum" and matter particles set rules of the game $\Longrightarrow \langle \Delta p \rangle_{vacuum} + \langle \Delta p \rangle_{matter} = 0$. The "vacuum turbulence" propels matter particles by transferring them an anti-Brownian **(recoil)** impulse (set $D = \hbar/2m$), whose "vacuum" trace (and reason) is the Brownian impulse (may die out, we track the matter data).

Step I. Given the **matter data** $\rho(x,t)$ and $v(x,t)$. At $t+\Delta t$ we have $\rho + \Delta\rho = \exp[-(\nabla v)\,\Delta t]\,\rho$ and $v \to v + \Delta v$, where:

Action ("vacuum" impulse)

$$\Delta v = +\frac{1}{m}\nabla(V+Q)\,\Delta t \text{ (Brownian)}$$

$$\textbf{is paralleled by: } (\Downarrow \textbf{ - subtract; } \Uparrow \textbf{ - add: } \frac{2}{m}\nabla(V+Q) \quad \textbf{!})$$

Reaction (matter impulse)

$$\Delta v = -\frac{1}{m}\nabla(V+Q)\,\Delta t \text{ (Anti-Brownian, e.g. quantum)}$$

Step II. **Update the matter data** to $\rho(x, t+\Delta t)$, $v(x, t+\Delta t)$, leave aside those referring to the "vacuum" and to the preceding Brownian impulse, **turn to the next** Δt episode when both impulses are excited **anew**.

Any **physical** justification of the **Brownian recoil principle needs a double-medium picture**:

(i) an active **"vacuum"** (background random field, non-equilibrium reservoir, zero-point fluctuations) that is generating and supporting Brownian impulses. These may be interpreted in terms **virtual particles**

(ii) **matter particles**, whose dynamics is governed by the III^{rd} Newton law and the resultant **recoil effect**.

A detailed theory of the "vacuum"-particle coupling is obviously necessary to go beyond heuristics.

There is plenty of room down there !

- atomic nucleus size: $\sim 10^{-15} - 10^{-14} m$
- atom size: $\sim 10^{-10} - 10^{-9} m$; what about its ψ-ness or that of the electron "cloud"?
- electron size (whatever that means): $\sim 10^{-15} m$, possibly down to $\sim 10^{-18} m$

Note: **The "vacuum" (not an empty void) functioning in quantum physics is still an open territory**