Probabilistic whereabouts of the „quantum potential”

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1. Major appearances of $\frac{\pm \Delta \rho^{1/2}}{\rho^{1/2}}$; Brownian vs quantum dynamics

2. Imaginary-time transformation: illusion of Euclidean time

3. Comments on variational extremum principles

4. Kinetic theory lore - analogies and hints

5. Heuristics of the Brownian recoil principle
\[ \int_R \rho(x) \, dx = 1, \text{ consider } -\ln \rho(x), -\nabla \ln \rho \text{ and } -\Delta \ln \rho \]

Shannon entropy of \( \rho \)

\[ S(\rho) = -\langle \ln \rho \rangle = -\int \rho(x) \ln \rho(x) \, dx \]

Fisher information measure of \( \rho \)

\[ \mathcal{F}(\rho) \doteq \langle (\nabla \ln \rho)^2 \rangle = \int \frac{(\nabla \rho)^2}{\rho} \, dx \]

(Isoperimetric) inequality:

\[ \mathcal{F} \geq (2\pi e) \exp(2S) \]

\[ \langle \nabla \ln x \rangle = 0 \text{ and } \text{Var}(x) = \sigma^2 = \langle (x - \langle x \rangle)^2 \rangle \longrightarrow \text{ an indeterminacy rule} \]

\[ \text{Var}(\nabla \ln x) = \mathcal{F}(\rho) \geq 1/\sigma^2 > 0 \]

Information theory note:

\[ \rho = |\psi|^2, \psi \in L^2 \longrightarrow \]

\[ (1/\sigma^2) \leq \mathcal{F} \leq 16\pi^2 \tilde{\sigma}^2 \quad \text{and} \quad (4\pi/\tilde{\sigma}) \leq (1/\sqrt{2\pi e}) \exp[S] \leq \sigma. \]
Playing with $-\ln \rho(x)$, $-\nabla \ln \rho$ and $-\Delta \ln \rho$, continued:

$$-\Delta \ln \rho = -\frac{\Delta \rho}{\rho} + \frac{(\nabla \rho)^2}{\rho^2} \implies -\langle \Delta \ln \rho \rangle = \langle \frac{(\nabla \rho)^2}{\rho^2} \rangle = \mathcal{F}(\rho).$$

For a potential of a "Newton-type force field" we have $\langle \nabla (\frac{\Delta \rho^{1/2}}{\rho^{1/2}}) \rangle = 0$ and

$$-\frac{\Delta \rho^{1/2}}{\rho^{1/2}} = \frac{1}{2}[-\frac{\Delta \rho}{\rho} + \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2}] \implies +\nabla (\frac{\Delta \rho^{1/2}}{\rho^{1/2}}) = \frac{1}{2\rho} \nabla (\rho \Delta \ln \rho)$$

$$-\langle \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \rangle = -\frac{1}{4} \langle \Delta \ln \rho \rangle = \frac{1}{4} \mathcal{F}(\rho) \geq \frac{1}{4Var(x)} > 0,$$

No indication of a specific physical context. However, a number of physically interesting quantities can be easily related with so-called local conservation laws for diffusion-type processes and the hydrodynamical formulation of the Schrödinger picture quantum dynamics.
Quantum hydrodynamics

\[ i\hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2m} \Delta + V \right] \psi \]

\( \hat{H} \) self-adjoint, \( \hat{H} \geq 0 \), \( \rho(x,t) = |\psi|^2(x,t) \), \( v = (\hbar/2mi)[(\nabla \psi/\psi) - (\nabla \psi^*/\psi^*)] \implies \)

\[ \partial_t \rho = -\nabla (\rho v); \quad \partial_t s + \frac{1}{2m} (\nabla s)^2 + (V + Q) = 0 \implies \]

\[ \partial_t v + (v \nabla v) = -\frac{1}{m} \nabla (V + Q) \]

where \( v = \frac{1}{m} \nabla s \) and \( Q = Q[\rho] = -\frac{\hbar^2}{2m} \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \).

Set \( |\psi| = \rho_*^{1/2} \). The ground state condition for \( \hat{H} \) reads: \( V = +\frac{\hbar^2}{2m} \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = -Q[\rho_*] \).

Denote \( u(x,t) \equiv (\hbar/2m) \nabla \ln \rho \). Dynamics arises via \{\rho, s\} extremum of \( I(\rho, s) = \int_{t_1}^{t_2} \left[ \partial_t s + \frac{m}{2} (u^2 + v^2) + V \right](t) dt \).

In terms of valid solutions \( \rho(x,t), s(x,t) \), we arrive at a strictly positive constant of motion: \(-\langle \partial_t s \rangle = H = \langle \left[ \frac{m}{2} (u^2 + v^2) + V \right] \rangle > 0 \) (finite energy condition).
Brownian hydrodynamics

$$\exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t \implies \partial_t\Psi = \left[D\Delta - \frac{V}{2mD}\right]\Psi$$

$\hat{H}$ self-adjoint, $\hat{H} \geq 0$, $t \geq 0$. Set $\hbar \equiv 2mD$. Let $\Psi(x, t) \to \rho_*^{1/2}$ as $t \to \infty$. Define $\rho(x, t) = \Psi(x, t)\rho_*^{1/2}(x)$ with $b = D\nabla \ln \rho_*$, $u = D\nabla \ln \rho$, $v = b - u = (1/m)\nabla s$.

$$\frac{V(x)}{2mD} = +D\frac{\Delta \rho_*^{1/2}}{\rho_*^{1/2}} = mD\left[\frac{b^2}{2D} + \nabla b\right] \implies$$

$$\partial_t \rho = D\Delta \rho - \nabla (b\rho) \iff \partial_t \rho = -\nabla (v\rho)$$

$$\partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0 \implies \partial_t v + (v\nabla v) = +\frac{1}{m}\nabla (V + Q)$$

Note a compatibility condition $V \equiv -Q[\rho_*]$. The \{\rho, s\} extremum principle for $I(\rho, s) = \int_{t_1}^{t_2} \langle \left[\partial_t s + (m/2)(v^2 - u^2) - V\right]\rangle$ yields equations of motion. On dynamically admitted fields $\rho(t)$ and $s(x, t)$, we have $-\langle \partial_t s \rangle = H = \langle \left[\frac{m}{2}(v^2 - u^2) - V\right]\rangle \equiv 0$!
Dynamical duality - illusion of "Euclidean time"

\[
\begin{align*}
it & \to t \geq 0; \quad \hbar \to 2mD \\
\exp(-i\hat{H}t/\hbar)\psi_0 &= \psi_t \longrightarrow \exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t
\end{align*}
\]

Given the spectral solution for \(\hat{H} = -\Delta + V\), the integral kernel of \(\exp(-t\hat{H})\) reads

\[
k(y, x, t) = \sum_j \exp(-\epsilon_j t) \Phi_j(y) \Phi_j^*(x)
\]

Remember that \(\epsilon_0 = 0\) and the sum may be replaced by an integral in case of a continuous spectrum, (with complex-valued generalized eigenfunctions). Set \(V(x) = 0\).

\[
k(y, x, t) = [\exp(t\Delta)](y, x) = (2\pi)^{-1/2} \int \exp(-p^2t) \exp(ip(y - x) \, dp = \\
(4\pi t)^{-1/2} \exp[-(y - x)^2/4t]
\]
Consider $\hat{H} = (1/2)(-\Delta + x^2 - 1)$ (e.g. rescaled harmonic oscillator Hamiltonian). The integral kernel of $\exp(-t\hat{H})$ is given by the classic Mehler formula:

$$k(y, x, t) = k(x, y, t) = \left[\exp(-t\hat{H})(y, x)\right] =$$

$$\left[\pi(1 - \exp(-2t))^{-1/2}\exp\left[-\frac{1}{2}(x^2 - y^2) - \frac{1}{2}(1 - \exp(-2t))^{-1}(x\exp(-t) - y)^2\right]\right]$$

The normalization condition $\int k(y, x, t) \exp\left[(y^2 - x^2)/2\right] dy = 1$ actually defines the transition probability density of the Ornstein-Uhlenbeck process

$$p(y, x, t) = k(y, x, t) \rho_\ast^{1/2}(x)/\rho_\ast^{1/2}(y)$$

with $\rho_\ast(x) = \pi^{-1/2}\exp(-x^2)$. A more familiar form of the kernel reads (note the presence of $\exp(t/2)$ factor)

$$k(y, x, t) = \frac{\exp(t/2)}{(2\pi \sinh t)^{1/2}} \exp\left[-\frac{(x^2 + y^2) \cosh t - 2xy}{2 \sinh t}\right]$$
Execute $t \to it$. We get a free Schrödinger propagator

$$K(y, x, t) = \left[ \exp(it\Delta) \right](y, x) = (2\pi)^{-1/2} \int \exp(-ip^2t) \exp(ip(y - x)) \, dp =$$

$$(4\pi it)^{-1/2} \exp[+i(y - x)^2/4t]$$

and likewise, that of (here $-1$ renormalized) harmonic oscillator propagator

$$K(y, x, t) = \frac{\exp(it/2)}{(2\pi i \sin t)^{1/2}} \exp \left[ +i \frac{(x^2 + y^2) \cos t - 2xy}{2 \sin t} \right]$$

Learn a standard Euclidean (field) theory lesson concerning multi-time correlation functions; exemplary harmonic oscillator case, $t > t' > 0$; $t \to it$:

$$E[X(t')X(t)] = \int \rho_*(x') \, x' \, p(x', t', x, t) \, x \, dz dx' = (1/2) \exp[-(t - t')] \implies$$

$$W(t', t) = \langle \psi_0, \hat{q}_H(t)\hat{q}_H(t')\psi_0 \rangle = (1/2) \exp[-i(t - t')]$$

Note: This appealing correspondence breaks down in $R^3$, in the presence of electromagnetic fields!
Comments on variational extremum principles

1. (Shannon) Entropy extremum principle: Given \( V = V(x) \), fix a priori \( \langle V \rangle = \zeta \). Extremize \( \mathcal{S} = -\langle \ln \rho \rangle \) under this constraint: seek an extremum of

\[
\mathcal{S}(\rho) + \alpha \langle V \rangle = \langle -\ln \rho + \alpha V \rangle
\]

where \( \alpha \) is a Lagrange multiplier. **Outcome:** \( \alpha \)-family of pdfs \( \rho_\alpha = A_\alpha \exp[\alpha V(x)] \) arises, provided \( (A_\alpha)^{-1} = \int \exp[\alpha V(x)] \, dx \) exists; \( \alpha \)-value comes from \( \langle V \rangle_\alpha = \zeta \).

2. Fisher information extremum principle: Fix a priori \( \langle V \rangle = \zeta \). Extremize the Fisher information measure \( \mathcal{F}(\rho) \) under that constraint:

\[
\mathcal{F}(\rho) + \lambda \langle V \rangle = \langle (\nabla \ln \rho)^2 + \lambda V \rangle
\]

Remember that \( -\langle \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \rangle = \frac{1}{4} \mathcal{F}(\rho) \). The extremizing pdf \( \rho(x) \doteq \rho_*(x) \) comes out from:

\[
V(x) = \frac{2}{\lambda} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] + \frac{4}{\lambda} \frac{\Delta \rho^{1/2}}{\rho^{1/2}}
\]

**Outcome:** \( \lambda \)-family of pdfs; \( \lambda \) gets fixed by \( \langle V \rangle_\lambda = \zeta \). Setting \( \lambda = 2/mD^2 \), we recover the **Brownian framework**; \( \lambda = 8m/\hbar^2 \) is admitted as a special case.
Hamilton-Jacobi route

\( H = \frac{p^2}{2m} + V(x), \{ \dot{q} = p/m, \dot{p} = -\nabla V(q) \}; \) assign \( \rho_0(x) \) (\( \Rightarrow S(\rho) \) and \( F(\rho) \)).

\[ I_0(\rho, s) = \int_{t_1}^{t_2} \left\langle \partial_t s + \frac{1}{2m} (\nabla s)^2 + V \right\rangle \, dt \implies \partial_t s + \frac{1}{2m} (\nabla s)^2 + V = 0 \]

and \( \partial_t \rho = -\nabla (v\rho) \). Here, \( v = (1/m)\nabla s \) and we have \( \partial_t v + (v\nabla v) = -\nabla V \).

Constrained Fisher information: Fix a priori \( \int_{t_1}^{t_2} F(\rho)(t) \, dt = \zeta \). Extremize

\[ I_\gamma(\rho, s) = \int_{t_1}^{t_2} dt \left\langle \left[ \partial_t s + \frac{(\nabla s)^2}{m} \pm V \right] + \gamma \frac{(\nabla \rho)^2}{\rho^2} \right\rangle \implies \]

\[ \partial_t \rho = -\nabla (v\rho) \]

\[ \partial_t s + \frac{(\nabla s)^2}{m} \pm V + 4\gamma \frac{\Delta \rho^{1/2}}{\rho^{1/2}} = 0 \]

\( \pm V \) is intended to make a distinction between confining and scattering potentials.
Outcomes (admissible case of $\gamma = 0$ is left aside):

(i) $\gamma = -mD^2/2$, eventually followed by setting $D = \hbar/2m$, leads to the $D$-labelled quantum hydrodynamics (before, we have referred to $+V$ only)

$$\partial_t s + \frac{1}{2m}(\nabla s)^2 \pm V + Q = 0$$

(ii) $\gamma = +mD^2/2$, with the potential term $-V$ only, leads to the Brownian hydrodynamics

$$\partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0$$

Note: $t \rightarrow it$ relationship can be secured for $+V$, where $V$ is a confining potential.

$$\partial_t s + \frac{1}{2m}(\nabla s)^2 + (V + Q) = 0$$

c.f. $t \rightarrow it \implies \exp(-t\hat{H}/2mD)\Psi_0 = \Psi_t \implies \exp(-it\hat{H}/2mD)\psi_0 = \psi_t$ issue.

We demand $\hat{H}$ to have a bottom eigenvalue equal zero (to yield a contractive semigroup). For a bounded from below Hamiltonian this can be always achieved, like e.g. in case of $\hat{H} = (1/2)(-\Delta + x^2 - 1)$. 
Hamilton-Jacobi route - a catalogue of "standards"

(i) $\mathcal{L}^+ = -\rho \left[ \partial_t s + (m/2)(v^2 + u^2) + V \right] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 + (V + Q) = 0$

(ii) $\mathcal{L}^\pm_{cl} = -\rho \left[ \partial_t s + (m/2)v^2 \pm V \right] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 \pm V = 0$

(iii) $\mathcal{L}^- = -\rho \left[ \partial_t s + (m/2)(v^2 - u^2) - V \right] \Rightarrow \partial_t s + (1/2m)(\nabla s)^2 - (V + Q) = 0$.

On dynamically admitted fields $\rho(t)$ and $s(x, t)$, $L(t) = \int dx \mathcal{L} \sim 0$, i.e. $\langle \partial_t s \rangle = -H$. The respective Hamiltonians obey:

(i) $H^+ = \int dx \rho \left[ (m/2)v^2 + V + (m/2)u^2 \right] > 0$, (quantum) constant of motion!

(ii) $H^\pm_{cl} = \int dx \rho \left[ (m/2)v^2 \pm V \right] = E, \quad E = (p^2/2m) \pm V(x)$, constant on each path

(iii) $H^- = \int dx \rho \left[ (m/2)v^2 - V - (m/2)u^2 \right] = 0$, identically (!) in Brownian motion

We emphasize that, from the start, $V(x)$ is chosen to be confining (a class of continuous and bounded from below functions allows to secure $\hat{H} \geq 0$).
Kinetic theory lore: Brownian analogies and hints

Consider free phase-space Brownian motion in the large friction regime. \(W(x, u, t)\) stands for phase-space (velocity-position) probability distribution with suitable initial data at \(t = 0\). Denote \(w(u, t)\) and \(w(x, t)\), the marginal pdfs. We set \(D = k_BT/m\beta\) and observe that actually, in the large friction regime, \(w(x, t) = (4\pi Dt)^{-1/2} \exp(-x^2/4Dt)\) solves \(\partial_t w = D\Delta w\).

\[
\langle u \rangle = \int du \ u W(x, u, t) \rightarrow \langle u \rangle = (x/2t)w(x, t) \\
\langle u \rangle_x = \langle u \rangle/w(x, t) = x/2t = -D(\nabla w)/w \\
\langle u^2 \rangle_x = \langle u^2 \rangle/w(x, t) = (D\beta - D/2t) + \langle u \rangle_x^2
\]

The Kramers-Fokker-Planck equation

\[
\partial_t W + u \nabla_x W = \beta \nabla_u (Wu) + q \Delta_u W
\]

with \(q = D\beta^2\), implies the local conservation laws

\[
\partial_t w + \nabla(\langle u \rangle_x w) = 0
\]

\[
\partial_t (\langle u \rangle_x w) + \nabla_x (\langle u^2 \rangle_x w) = -\beta \langle u \rangle_x w
\]
Introducing the kinetic pressure $P_{\text{kin}}(x, t) = [\langle u^2 \rangle_x - \langle u \rangle_x^2]w(x, t)$ we arrive at
\[
\partial_t + \langle u \rangle_x \nabla \langle u \rangle_x = -\beta \langle u \rangle_x - \nabla P_{\text{kin}}/w
\]
In the large friction regime we have
\[
-\frac{\nabla P_{\text{kin}}}{w} = +\beta \langle u \rangle_x - \frac{\nabla P_{\text{osm}}}{w}
\]
where $P_{\text{osm}} = D^2 w \Delta \ln w$ we name an osmotic pressure in the Brownian motion.

\[
\nabla P_{\text{osm}} = -w \nabla Q/m \quad \text{with} \quad Q = -2mD^2 \Delta w^{1/2}/w^{1/2}
\]

Actually $-\nabla P_{\text{osm}} = (D/2t) \nabla w$. Thus, denoting $\langle u \rangle_x = v(x, t)$ we arrive at:
\[
(\partial_t + v \nabla) v = -\frac{\nabla P_{\text{osm}}}{w} = +\frac{1}{m} \nabla Q
\]
to be compared with the general Brownian hydrodynamics result
\[
\partial_t v + (v \nabla v) = +\frac{1}{m} \nabla (V + Q)
\]
In the past (1992) I have named all that: "derivation of the quantum potential from realistic Brownian particle motions".
Concerning the pressure terms $P_{\text{kin}}$ and $P_{\text{osm}}$

In view of $-\langle \Delta \ln \rho \rangle = \mathcal{F}(\rho) > 0$, $P_{\text{osm}}$ is predominantly negative-definite. To the contrary, $P_{\text{kin}}$ is positive definite, hence the large friction regime is valid for times $t > (2\beta)^{-1}$. Let us introduce the kinetic temperature:

$$0 \leq \Theta_{\text{kin}} = m \frac{P_{\text{kin}}}{w} \sim (k_B T - \frac{mD}{2t}) < k_B T$$

whose (large time limit) asymptotic value, $k_B T$ actually is. Since $P_{\text{osm}}/w = D^2 \Delta \ln w = -D/2t$, we learn that a (predominantly !) positive-definite quantity

$$\Theta_{\text{osm}} = -m \frac{P_{\text{osm}}}{w} = -mD^2 \Delta \ln w \implies \Theta_{\text{kin}} \sim (k_B T - \Theta_{\text{osm}})$$

gives account of the deviation from thermal equilibrium in terms of the local ”thermal energy” (agitation) $\Theta_{\text{osm}}$.

One more useful identity (not an independent equation) is valid. It expresses the ”thermal energy” conservation law (no thermal currents are hereby induced):

$$(\partial_t + u \nabla) \Theta_{\text{osm}} = -2(\nabla u) \Theta_{\text{osm}} \implies \partial_t \Theta_{\text{osm}} = -2(\nabla u) \Theta_{\text{osm}}$$
Meaning of the pressure term in Brownian hydrodynamics \((P_{osm} \equiv P)\)

\[
\partial_t v + (v \nabla v) = +\frac{1}{m} \nabla (V + Q) = \frac{1}{m} F - \frac{\nabla P}{w}; \quad -\frac{\nabla P}{w} = +\frac{1}{m} \nabla Q; \quad F \equiv -\nabla (-V)
\]

In normal liquids the pressure is exerted upon any control volume (droplet) \(\Rightarrow\) a compression of a droplet. In case of Brownian motion, we deal with a definite decompression.

Consider a reference volume (control interval, finite droplet) \([\alpha, \beta]\) in \(R^1\) (or \(\Lambda \subset R^1\)) which at time \(t > 0\) comprises a certain fraction of particles (Brownian "fluid" constituents).

The time rate of particles loss or gain by the volume \([\alpha, \beta]\) at time \(t\), is equal to the flow outgoing through the boundaries

\[
-\partial_t \int_{\alpha}^{\beta} \rho(x, t) dx = \rho(\beta, t)v(\beta, t) - \rho(\alpha, t)v(\alpha, t)
\]

To analyze the momentum balance, let us slightly deform the boundaries \([\alpha, \beta]\) to compensate the mass imbalance: \([\alpha, \beta]\) \(\rightarrow [\alpha + v(\alpha, t)\Delta t, \beta + v(\beta, t)\Delta t]\). Effectively, we pass to a locally co-moving (droplet) frame (that is the Lagrangian picture).
(i) The mass balance in the moving droplet has been achieved:

\[
\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int_{\alpha+\Delta t}^{\beta+v_{\beta} \Delta t} \rho(x, t + \Delta t) dx - \int_{\alpha}^{\beta} \rho(x, t) dx \right] = 0
\]

(ii) For local matter flows \((\rho v)(x, t)\), in view of \(\partial_t (\rho v) = -\nabla (\rho v^2) + (1/m) \rho \nabla (V + Q)\), the rate of change of momentum (per unit of mass) of the droplet, reads

\[
\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int_{\alpha+\Delta t}^{\beta+v_{\beta} \Delta t} (\rho v)(x, t + \Delta t) - \int_{\alpha}^{\beta} (\rho v)(x, t) \right] = \int_{\alpha}^{\beta} \frac{1}{m} \nabla (V + Q) dx
\]

However, \(\nabla Q/m = -\frac{\nabla P}{\rho}\) and \(P = D^2 \rho \Delta ln \rho\). Therefore:

\[
\int_{\alpha}^{\beta} \frac{1}{m} \nabla (V + Q) dx = \int_{\alpha}^{\beta} \rho \nabla \Omega dx - \int_{\alpha}^{\beta} \nabla P dx = \frac{1}{m} E [\nabla V]_{\alpha}^{\beta} + P(\alpha, t) - P(\beta, t)
\]

(iii) The time rate of change of the kinetic energy of the droplet is:

\[
\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} \left[ \int_{\alpha+\Delta t}^{\beta+v_{\beta} \Delta t} \frac{1}{2} (\rho v^2)(x, t + \Delta t) - \int_{\alpha}^{\beta} \frac{1}{2} (\rho v^2)(x, t) \right] = \int_{\alpha}^{\beta} \frac{1}{m} (\rho v) \nabla (V + Q) dx
\]

Note that \(\int_{\alpha}^{\beta} \rho v \nabla Q dx = -\int_{\alpha}^{\beta} v \nabla P dx\) (c.f. the notion of power release \(\frac{dE}{dt} = F \cdot v\))
Meaning of the pressure term in quantum hydrodynamics \(( -P_{osm} = P)\)

\[
\partial_t v + (v \nabla v) = - \frac{1}{m} \nabla (V + Q) = \frac{F}{m} - \frac{\nabla P}{\rho} \implies \\
- \frac{1}{m} \nabla Q = + \frac{\nabla P_{osm}}{\rho} \equiv - \frac{\nabla P}{\rho}
\]

which enforces \(-P_{osm} = -D^2 \rho \Delta \ln \rho = P\), \(D = \hbar/2m\), while \(F = -\nabla V\). If compared to the Brownian hydrodynamics all \((V + Q)\) contributions come with an inverted sign. That carries over to the mass, momentum and kinetic energy rates. Contrary to the Brownian \(P = P_{osm}\), the quantum pressure term \(P = -P_{osm}\) is predominantly positive. We recall that \(-\langle \Delta \ln \rho \rangle = \langle \frac{(\nabla \rho)^2}{\rho^2} \rangle = \mathcal{F}(\rho) > 0\).

We note in passing that quantum mechanically derivable heat transfer equation

\[
(\partial_t + v \nabla) \Theta_{osm} = -2 \frac{\nabla q}{\rho} - 2(\nabla v) \Theta_{osm}
\]

with \(\Theta_{osm} = -m \frac{P_{osm}}{\rho} = -mD^2 \Delta \ln \rho\) and \(q = -2mD^2 \rho \Delta v\), reproduces the Brownian form, at least for generic free Schrödinger wave packets with \(\Delta v = 0\). We get \(\partial_t \Theta_{osm} = -2(\nabla v) \Theta_{osm}\) as well. There is no heat current.
Hamilton-Jacobi related hydrodynamics and (Bohmian) trajectory descriptions

Eulerian picture (passive control) vs Lagrangian picture (active control in a co-moving frame): simply give our previous droplet (co-moving control volume) an infinitesimal size. We get droplet dynamics along Bohm-type trajectories.

\[ f(x, t) \to f(x(t + \Delta t), t + \Delta t) \sim [\partial_t f + (v \nabla) f] \Delta t; \quad \dot{x} = v = v(x, t)|_{x(t)=x} \]

\[ x(t + \Delta t) \sim v\Delta t, \quad v = (1/m)\nabla s \text{ and } \partial_t s = \frac{ds}{dt} - mv^2 \text{ imply} \]

(i) Classical hydrodynamics: (droplet) paths in the Lagrangian frame

\[ \frac{d\rho}{dt} = -\rho \nabla v \quad \longrightarrow \quad \rho(x(t + \Delta t), t + dt) \sim \exp[-(\nabla v)\Delta t] \rho(x, t) \]

\[ \frac{ds}{dt} = \frac{1}{2m}(\nabla s)^2 - (\pm V) \quad \Longrightarrow \quad m \frac{dv}{dt} = -\nabla(\pm V) \]
(ii) Brownian hydrodynamics: (droplet) paths in the Lagrangian frame

\[ \frac{d\rho}{dt} = -\rho \nabla v \]

\[ \frac{ds}{dt} = \frac{1}{2m} (\nabla s)^2 + (V + Q) \implies m \frac{dv}{dt} = +\nabla (V + Q) \]

Purely random (Wiener) background: \( dX(t) = b(X(t))dt + \sqrt{2D}dW(t) \implies \dot{\rho} = D \Delta \rho - \nabla (b\rho); \quad \frac{\psi(x)}{2mD} = mD \left[ \frac{\dot{b}^2}{2D} + \nabla b \right] \]

(iii) Quantum hydrodynamics: (droplet) paths in the Lagrangian frame \implies Bohmian trajectories

\[ \frac{d\rho}{dt} = -\rho \nabla v \]

\[ \frac{ds}{dt} = \frac{1}{2m} (\nabla s)^2 - (V + Q) \implies m \frac{dv}{dt} = -\nabla (V + Q) \]
Back to random paths: diffusion-type processes

Consider a Markovian diffusion process on $R$, for times $t \in [0,T]$: $dX(t) = b(X(t), t)dt + \sqrt{2DdW(t)}$, where $W(t)$ stands for the Wiener noise and $X(t_0) = x_0$. Given $p(y, s, x, t), s \leq t$ and $\rho_0(x)$, we can infer a statistical future of the process:

$$\rho(x, t) = \int \rho(y, s)p(y, s, x, t) dy \rightarrow \partial_t \rho = D\Delta - (\nabla b \rho)$$

$$b(x, t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta t} \int (y - x)p(x, t, y, t + \Delta t) dy = v(x, t) + (D\nabla \rho / \rho)(x, t)$$

We can as well reproduce a statistical past of the process, by means of

$$p_*(y, s, x, t) = p(y, s, x, t)\frac{\rho(y, s)}{\rho(x, t)} \Rightarrow \rho(y, s) = \int p_*(y, s, x, t)\rho(x, t) dx$$

$$b_*(y, s) = \lim_{\Delta s \rightarrow 0} \frac{1}{\Delta s} \int (y - s)p_*(x, s - \Delta s, y, s) dx = v(y, s) - (D\nabla \rho / \rho)(y, s)$$

Making notice of $v = (1/2)(b + b_*)$, we get:

$$\partial_t \rho = -\nabla (v \rho) = D\Delta \rho - (\nabla b \rho) = -D\Delta \rho - \nabla (b_* \rho)$$
**Incoming random flow**

**Outgoing random flow**

\[ b_*(X(t - \Delta t), t - \Delta t) \]

\[ \langle b_* \rangle(x, t - \Delta t) \]

\[ b_*(x, t) \]

\[ b(X(t + \Delta t), t + \Delta t) \]

\[ \langle b \rangle(x, t + \Delta t) \]

\[ b(x, t) \]

**Impulsive behavior of drifts in Brownian motion**

\[ b_*(x, t) - \langle b_* \rangle(x, t - \Delta t) \sim \langle b \rangle(x, t + \Delta t) - b(x, t) \sim \frac{1}{m} \nabla V \Delta t \]

**Impulsive behavior of drifts in stochastic mechanics**

\[ b_*(x, t) - \langle b_* \rangle(x, t - \Delta t) \sim \langle b \rangle(x, t + \Delta t) - b(x, t) \sim \frac{1}{m} \nabla (V + 2Q) \Delta t \]
Consider $b = DX$ and $b_\star = D_\star X$ as special cases of forward (predictive) and backward (retrodictive) time derivatives of functions of the random variable $X(t)$:

$$(Df)(X(t), t) = (\partial_t + b \nabla + D\Delta)f; \quad (D_\star f)(X(t), t) = (\partial_t + b_\star \nabla - D\Delta)f$$

Analyze acceleration formulas for diffusion-type processes.

**II$^{nd}$** Newton law in the (local) mean

(i) Brownian motion

$$(D^2 X)(t) = (\partial_t + v \nabla)v - \frac{1}{m} \nabla Q = (D^2_\star X)(t) = +\frac{1}{m} \nabla V$$

(ii) Nelson’s stochastic mechanics

$$\frac{1}{2}[(DD_\star + D_\star D)X](t) = (\partial_t + v \nabla)v + \frac{1}{m} \nabla Q = -\frac{1}{m} \nabla V$$

Something is conspicuously missing: set $\nabla V = 0$, still accelerating! $\implies$

$$(i) \quad \frac{dv}{dt} - \frac{1}{m} \nabla Q = 0, \quad (ii) \quad \frac{dv}{dt} + \frac{1}{m} \nabla Q = 0$$

$\mp \nabla Q$ contributes to the $\pm \Delta v$ velocity increment as a legitimate force.

Our preference for the II$^{nd}$ Newton law is:

(iii) $$(\partial_t + v \nabla)v = \pm \frac{1}{m} \nabla(Q + V)$$
Brownian impulse (times $\Delta t$)

$$(D^2X)(t) = (D^2_*X)(t) = +\frac{1}{m} \nabla V \iff \frac{1}{2}[(DD_* + D_*D)X](t) = \frac{1}{m} \nabla (V + 2Q)$$

Stochastic mechanics impulse (times $\Delta t$)

$$(D^2X)(t) = (D^2_*X)(t) = -\frac{1}{m} \nabla (V + 2Q) \iff \frac{1}{2}[(DD_* + D_*D)X](t) = -\frac{1}{m} \nabla V$$

$\downarrow$  III$^{rd}$ Newton law in the mean: $\pm \frac{2}{m} \nabla (V + Q)$  $\uparrow$

Impulse-momentum change equations

Brownian impulse in a co-moving frame (given $\rho$ and $v$)

$$\Delta \rho = -[(\nabla v) \Delta t] \rho \quad m \Delta v = +\nabla (V + Q) \Delta t$$

Anti-Brownian impulse in a co-moving frame (given $\rho$ and $v$)

$$\Delta \rho = -[(\nabla v) \Delta t] \rho \quad m \Delta v = -\nabla (V + Q) \Delta t$$
Introducing the „Brownian recoil principle”

C.f. physics of firearms (note a recoiless gun); I am sorry for military associations
Brownian recoil principle

Consider $\Delta t \ll 1$. Within $[t, t + \Delta t]$, let the action-reaction coupling between "vacuum" and matter particles set rules of the game $\implies \langle \Delta p \rangle_{\text{vacuum}} + \langle \Delta p \rangle_{\text{matter}} = 0$. The "vacuum turbulence" propels matter particles by transferring them an anti-Brownian (recoil) impulse (set $D = \hbar/2m$), whose "vacuum" trace (and reason) is the Brownian impulse (may die out, we track the matter data).

Step I. Given the matter data $\rho(x, t)$ and $v(x, t)$. At $t + \Delta t$ we have $\rho + \Delta \rho = \exp[-(\nabla v) \Delta t] \rho$ and $v \rightarrow v + \Delta v$, where:

**Action ("vacuum" impulse)**

$$\Delta v = +\frac{1}{m} \nabla (V + Q) \Delta t \text{ (Brownian)}$$

is paralleled by: $(\downarrow - \text{subtract}; \uparrow - \text{add}: \frac{2}{m} \nabla (V + Q)$ !

**Reaction (matter impulse)**

$$\Delta v = -\frac{1}{m} \nabla (V + Q) \Delta t \text{ (Anti-Brownian, e.g. quantum)}$$

Step II. Update the matter data to $\rho(x, t + \Delta t)$, $v(x, t + \Delta t)$, leave aside those referring to the "vacuum" and to the preceding Brownian impulse, turn to the next $\Delta t$ episode when both impulses are excited anew.
Any physical justification of the Brownian recoil principle needs a double-medium picture:

(i) an active "vacuum" (background random field, non-equilibrium reservoir, zero-point fluctuations) that is generating and supporting Brownian impulses. These may be interpreted in terms virtual particles

(ii) matter particles, whose dynamics is governed by the IIIrd Newton law and the resultant recoil effect.

A detailed theory of the "vacuum"-particle coupling is obviously necessary to go beyond heuristics.

There is plenty of room down there!
- atomic nucleus size: $\sim 10^{-15} - 10^{-14} m$
- atom size: $\sim 10^{-10} - 10^{-9} m$; what about its $\psi$-ness or that of the electron "cloud"?
- electron size (whatever that means): $\sim 10^{-15} m$, possibly down to $\sim 10^{-18} m$

Note: The "vacuum" (not an empty void) functioning in quantum physics is still an open territory